HIGHER APR TILTING PRESERVE n-REPRESENTATION INFINITENESS

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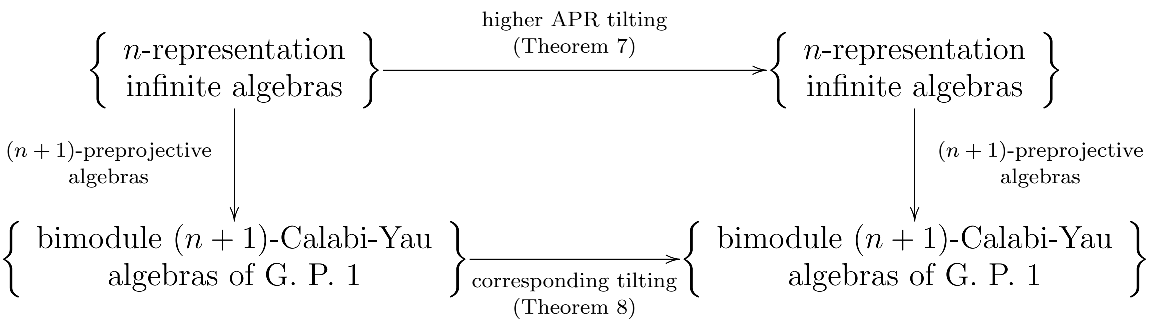
Abstract. We show that m-APR tilting preserves n-representation infiniteness for $1 \leq m \leq n$. Moreover, we show that these tilting modules lift to tilting modules for the corresponding higher preprojective algebras, which is $(n+1)$-CY algebras. We also study the interplay of the two kinds of tilting modules.

1. Introduction

In this note, we show that m-APR tilting modules preserve n-representation infiniteness for m with $1 \leq m \leq n$. By this fact, we obtain a large family of n-representation infinite algebras. Our next result is that these modules lift to tilting modules over the corresponding $(n+1)$-preprojective algebras. Moreover, we show that the $(n+1)$-preprojective algebra of an m-APR tilted algebra is isomorphic to the endomorphism algebra of the corresponding tilting module induced by the m-APR tilting module. This also implies that we obtain a family of $(n+1)$-CY algebras, which are derived equivalent to each other.



Notations. Let K be an algebraically closed field. We denote by $D := \mathrm{Hom}_K(-, K)$ the K-dual. An algebra means a K-algebra which is indecomposable as a ring. For an algebra Λ, we denote by $\mathrm{Mod}\,\Lambda$ the category of right Λ-modules and by $\mathrm{mod}\,\Lambda$ the category of finitely generated Λ-modules. If Λ is $\mathbb{Z}$-graded, we denote by $\mathrm{Mod}^{\mathbb{Z}}\,\Lambda$ the category of $\mathbb{Z}$-graded Λ-modules and by $\mathrm{mod}^{\mathbb{Z}}\,\Lambda$ the category of finitely generated $\mathbb{Z}$-graded Λ-modules.

2. Preliminaries

2.1. n-representation infinite algebras.

Let Λ be a finite dimensional algebra of global dimension at most n. We let $\mathcal{D}^{\mathrm{b}}(\Lambda) := \mathcal{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$ and denote the Nakayama functor by

$$\nu := - \otimes^{\mathbb{L}}_{\Lambda} D\Lambda \simeq D\,\mathbb{R}\mathrm{Hom}_{\Lambda}(-, \Lambda) : \mathcal{D}^{\mathrm{b}}(\Lambda) \longrightarrow \mathcal{D}^{\mathrm{b}}(\Lambda).$$

The detailed version of this paper will be submitted for publication elsewhere.

Then ν gives a Serre functor, i.e. there exists a functorial isomorphism

$$\mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(\Lambda)}(X, Y) \simeq D\,\mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(\Lambda)}(Y, \nu X)$$

for any $X, Y \in \mathcal{D}^{\mathrm{b}}(\Lambda)$. A quasi-inverse of ν is given by

$$\nu^{-} := \mathbb{R}\mathrm{Hom}_{\Lambda}(D\Lambda, -) \simeq - \otimes^{\mathbb{L}}_{\Lambda} \mathbb{R}\mathrm{Hom}_{\Lambda}(D\Lambda, \Lambda) : \mathcal{D}^{\mathrm{b}}(\Lambda) \longrightarrow \mathcal{D}^{\mathrm{b}}(\Lambda).$$

We let

$$\nu_n := \nu \circ [-n] \text{ and } \nu_n^{-} := \nu^{-} \circ [n].$$

Then we recall the definition of n-representation infinite algebras as follows.

Definition 1. [4] A finite dimensional algebra Λ is called *n-representation infinite* if it satisfies $\mathrm{gl.dim}\,\Lambda \leq n$ and $\nu_n^{-i}(X) \in \mathrm{mod}\Lambda$ for any $i \geq 0$.

2.2. **m-APR tilting modules.** Let Λ be an algebra. A Λ-module T is called *tilting* if it satisfies the following conditions.

(T1) There exists an exact sequence

$$0 \to P_m \to \cdots\cdots P_1 \to P_0 \to T \to 0$$

where each P_i is a finitely generated projective Λ-module.

(T2) $\mathrm{Ext}^i_{\Lambda}(T, T) = 0$ for any $i > 0$.

(T3) There exists an exact sequence

$$0 \to \Lambda \to T_0 \to T_1 \to \cdots\cdot \to T_m \to 0$$

where each T_i belongs to $\mathrm{add}\, T$.

In this case, there exists a triangle-equivalence between $\mathcal{D}^{\mathrm{b}}(\Lambda)$ and $\mathcal{D}^{\mathrm{b}}(\mathrm{End}(T))$.

The following tilting modules play a central role in this paper.

Definition 2. Let Λ be a finite dimensional algebra of global dimension at most n. We assume that there is a simple projective Λ-module S satisfying $\mathrm{Ext}^i_{\Lambda}(D\Lambda, S) = 0$ for any $1 \leq i < n$. Take a direct sum decomposition $\Lambda = S \oplus Q$ as a Λ-module. In [5, Proposition 3.2] (and its proof), it was shown that there exists a minimal projective resolution

$$0 \to S \xrightarrow{a_0} P_1 \xrightarrow{a_1} \cdots\cdots \xrightarrow{a_{n-1}} P_n \xrightarrow{a_n} \tau_n^{-}(S) \to 0$$

of $\tau_n^{-}(S)$ such that each P_i belongs to $\mathrm{add}\, Q$. Let $K_m := \mathrm{Im}\, a_m$ for $0 \leq m \leq n$. Note that $K_0 = S$ and $K_n = \tau_n^{-}(S)$. Then it was shown that $Q \oplus K_m$ is a tilting module with projective dimension m. Following [5], we call it the *m-APR* (=Auslander-Platzeck-Reiten) tilting module with respect to S.

If Λ is an n-representation infinite algebra with a simple projective module S, then the above condition $\mathrm{Ext}^i_{\Lambda}(D\Lambda, S) = 0$ $(1 \leq i < n)$ is automatically satisfied. Thus any simple projective module gives an m-APR tilting module for n-representation infinite algebras.

2.3. $(n+1)$**-preprojective algebras.** Next we recall the definition of $(n+1)$-preprojective algebras and their property. In the case of $n = 1$, the algebras coincide with the (classical) preprojective algebras.

Definition 3. [6] Let Λ be a finite dimensional algebra. The $(n+1)$*-preprojective algebra* $\widehat{\Lambda}$ for Λ is a tensor algebra

$$\widehat{\Lambda} := T_\Lambda(\mathrm{Ext}^n_\Lambda(D\Lambda, \Lambda))$$

of $\Lambda^{\mathrm{op}} \otimes_K \Lambda$-module $\mathrm{Ext}^n_\Lambda(D\Lambda, \Lambda)$. This algebra can be regarded as a positively graded algebra by

$$\widehat{\Lambda}_i = \mathrm{Ext}^n_\Lambda(D\Lambda, \Lambda)^{\otimes^i_\Lambda} = \overbrace{\mathrm{Ext}^n_\Lambda(D\Lambda, \Lambda) \otimes_\Lambda \cdots\cdots \otimes_\Lambda \mathrm{Ext}^n_\Lambda(D\Lambda, \Lambda)}^{i}.$$

We remark that the $(n+1)$-preprojective algebra is the 0-th homology of Keller's *derived* $(n+1)$*-preprojective DG algebra* [8].

Moreover, we recall the following definition, which is a graded analog of Ginzburg's Calabi-Yau algebras.

Definition 4. Let $A = \bigoplus_{i \geq 0} A_i$ be a positively graded algebra such that $\dim_K A_i < \infty$ for any $i \geq 0$. We denote by $A^e := A^{\mathrm{op}} \otimes_K A$. We call A *bimodule n-Calabi-Yau of Gorenstein parameter* 1 if it satisfies the following conditions.

(1) $A \in \mathcal{K}^{\mathrm{b}}(\mathrm{proj}^{\mathbb{Z}} A^e)$.
(2) $\mathbb{R}\mathrm{Hom}_{A^e}(A, A^e)[n](-1) \simeq A$ in $\mathcal{D}(\mathrm{Mod}^{\mathbb{Z}} A^e)$.

Then n-representation infinite algebras and bimodule CY algebras have a close relationship as follows (see [4, Theorem 4.35]).

Theorem 5. [1, 8, 9] *There is a one-to-one correspondence between isomorphism classes of n-representation infinite algebras Λ and isomorphism classes of graded bimodule $(n+1)$-CY algebras A of Gorenstein parameter* 1*. The correspondence is given by*

$$\Lambda \mapsto \widehat{\Lambda} \quad \textit{and} \quad A \mapsto A_0.$$

The following result implies that bimodule n-CY algebras provide n-CY triangulated categories.

Theorem 6. [7, Lemma 4.1][3, Proposition 3.2.4] *Let A be a bimodule n-CY algebra. Then there exists a functorial isomorphism*

$$\mathrm{Hom}_{\mathcal{D}(\mathrm{Mod}\, A)}(M, N) \simeq D\,\mathrm{Hom}_{\mathcal{D}(\mathrm{Mod}\, A)}(N, M[n])$$

for any $N \in \mathcal{D}(\mathrm{Mod}\, A)$ whose total homology is finite dimensional and any $M \in \mathcal{D}(\mathrm{Mod}\, A)$.

3. Our results

Let Λ be an n-representation infinite algebra. Assume that there exists a simple projective Λ-module S and take a direct sum decomposition $\Lambda = S \oplus Q$ as a Λ-module. As Definition 2, we have a minimal projective resolution

$$0 \to S \xrightarrow{a_0} P_1 \xrightarrow{a_1} \cdots\cdots \xrightarrow{a_{n-1}} P_n \xrightarrow{a_n} \tau_n^-(S) \to 0 \tag{3.1}$$

of $\tau_n^-(S)$ such that each P_i belongs to $\operatorname{add} Q$. Let $K_i := \operatorname{Im} a_i$ and we fix m with $0 \leq m \leq n$. Then we denote, respectively, the m-APR tilting Λ-module and the endomorphism algebra by

$$T := Q \oplus K_m \qquad \text{and} \qquad \Gamma := \operatorname{End}_\Lambda(T). \tag{3.2}$$

Our first result is the following one, which is a generalization of [4, Theorem 2.13].

Theorem 7. *Under the above setting, the algebra Γ is n-representation infinite.*

Moreover we show that m-APR tilting modules over n-representation infinite algebras lift to tilting modules over the corresponding $(n+1)$-preprojective algebras.

Let $\widehat{\Lambda} := \bigoplus_{i \geq 0} \widehat{\Lambda}_i$ and $\mathcal{D}(\widehat{\Lambda}) := \mathcal{D}(\operatorname{Mod} \widehat{\Lambda})$. For a $\mathbb{Z}$-graded $\widehat{\Lambda}$-module X, we write X_ℓ the degree ℓ-th part of X. For a $\mathbb{Z}$-graded finitely generated $\widehat{\Lambda}$-module X, the algebra $\operatorname{End}_{\widehat{\Lambda}}(X)$ can be regarded as a $\mathbb{Z}$-graded algebra by $\operatorname{End}_{\widehat{\Lambda}}(X)_i = \operatorname{Hom}_{\widehat{\Lambda}}(X, X(i))_0$, where (i) is a graded shift functor and $\operatorname{Hom}_{\widehat{\Lambda}}(X, X)_0 := \{f \in \operatorname{Hom}_{\widehat{\Lambda}}(X, X) \mid f(X_i) \subset X_i \text{ for any } i\}$.

Moreover, an algebra $\Lambda^{\mathrm{op}} \otimes_K \widehat{\Lambda}$ can be regarded as a $\mathbb{Z}$-graded algebra by $(\Lambda^{\mathrm{op}} \otimes_K \widehat{\Lambda})_i := \Lambda^{\mathrm{op}} \otimes_K (\widehat{\Lambda})_i$. Thus we regard $\widehat{\Lambda}$ as a $\mathbb{Z}$-graded $(\Lambda^{\mathrm{op}} \otimes_K \widehat{\Lambda})$-module and we have a functor

$$\widehat{(\ )} := - \otimes_\Lambda \widehat{\Lambda} : \operatorname{mod} \Lambda \longrightarrow \operatorname{mod}^{\mathbb{Z}} \widehat{\Lambda}.$$

Note that we have

$$\widehat{X}_i = \begin{cases} 0 & (i \leq 0) \\ \tau_n^{-i}(X) & (i \geq 0) \end{cases}$$

for any $X \in \operatorname{mod} \Lambda$. Then we obtain the following results.

Theorem 8. *Under the above setting, the following assertions hold.*

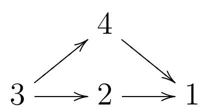
(1) *$\widehat{T}$ is a tilting $\widehat{\Lambda}$-module of projective dimension m.*
(2) *$\operatorname{End}_{\widehat{\Lambda}}(\widehat{T})$ is isomorphic to the (n+1)-preprojective algebra $\widehat{\Gamma}$ of Γ. In particular, $\operatorname{End}_{\widehat{\Lambda}}(\widehat{T})$ is a graded bimodule (n+1)-CY algebra of Gorenstein parameter 1.*

For the case of $m = n$, m-APR tilting modules have a particularly nice property as stated below.

Corollary 9. *Assume that T is an n-APR tilting Λ-module. Then there exists an isomorphism $\widehat{\Lambda} \simeq \widehat{\Gamma}$ of algebras.*

Example 10.

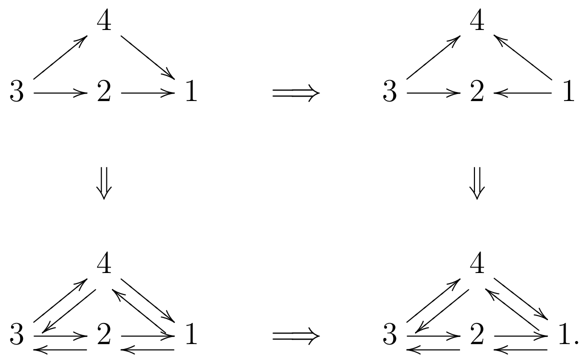
(1) First we give an example for the classical case, namely the case of $n = m = 1$. Let Q be the following quiver.



We consider the path algebra $\Lambda := KQ$ of Q, which is 1-representation infinite, and the 1-APR tilting Λ-module T associated with vertex 1. Then $\Gamma := \operatorname{End}_\Lambda(T)$ is also a 1-representation infinite algebra, which is the path algebra of the quiver obtained from Q by reversing the arrows ending at the vertex 1.

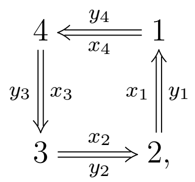
It is known that the 2-preprojetive algebras $\widehat{\Lambda}$ and $\widehat{\Gamma}$ are given by the double quiver of the quiver of Λ and Γ with some relations respectively. Moreover T induces a tilting $\widehat{\Lambda}$-module $\widehat{T}$ with $\widehat{\Gamma} \simeq \mathrm{End}_{\widehat{\Lambda}}(\widehat{T}) \simeq \widehat{\Lambda}$ by Theorem 8 and Proposition 9.

These results imply the compatibility of the following diagram of quivers, where horizontal arrows indicate tilts of T and $\widehat{T}$, respectively, and vertical arrows indicate taking 2-preprojective algebras.



(2) Next we give an example for the case $m = 1 < 2 = n$. We note that the structure of 3-CY algebras has been extensively studied and it is known that they have a close relationship with quivers with potentials (QPs).

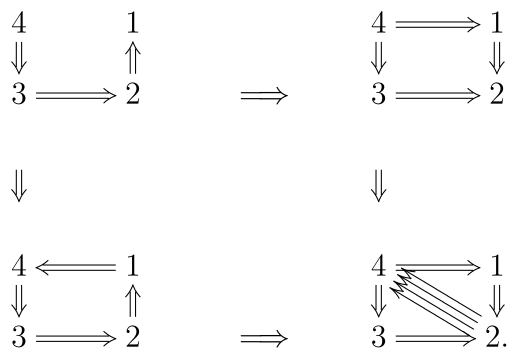
Let Q be a quiver



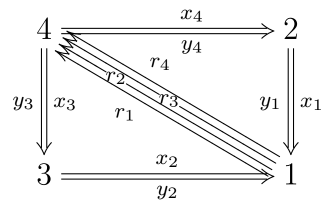
and $W := x_1x_2x_3x_4 - y_1y_2y_3y_4 + x_1y_2x_3y_4 - y_1x_2y_3x_4$ a *potential* on Q and $C := \{x_4, y_4\}$ a *cut*. Then the *truncated Jacobian algebra* Λ of (Q, W, C) is a 2-representation infinite algebra (see [1, section 6]), whose quiver is the left upper one in the picture below. We can consider the 1-APR tilting Λ-module T associated with vertex 1. By Theorem 7, $\Gamma := \mathrm{End}_\Lambda(T)$ is also a 2-representation infinite algebra. Moreover T induces a tilting $\widehat{\Lambda}$-module $\widehat{T}$ with $\widehat{\Gamma} \simeq \mathrm{End}_{\widehat{\Lambda}}(\widehat{T})$.

In this example, we can understand the change of quivers with relations of tilts and the 3-preprojective algebras. Indeed, it is known that the quiver with relations of Γ can be calculated by applying mutation of graded QPs. On the other hand, the 3-preprojective algebra $\widehat{\Lambda}$ is given as the *Jacobian algebra* of (Q, W) (see [8]), and $\mathrm{End}_{\widehat{\Lambda}}(\widehat{T})$ is given as the Jacobian algebra of the QP obtained by mutating (Q, W).

Therefore, we have the following diagram of quivers, where horizontal arrows indicate tilts of T and $\widehat{T}$, respectively, and vertical arrows indicate taking 3-preprojective algebras.



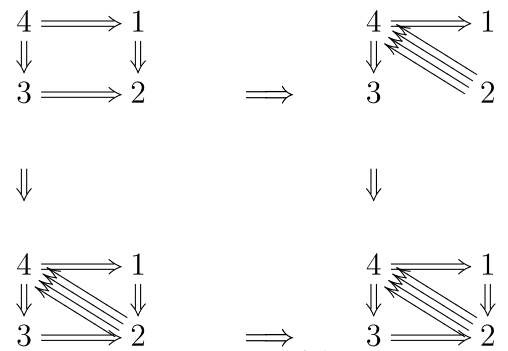
(3) Finally we give an example for the case $n = m = 2$. Let Q be a quiver



and $W := (x_1x_4 - x_2x_3)r_1 + (x_1y_4 - y_2x_3)r_2 + (y_1x_4 - x_2y_3)r_3 + (y_1y_4 - y_2y_3)r_4$ a potential on Q and $C := \{r_1, r_2, r_3, r_4\}$ a cut. Then the truncated Jacobian algebra Λ of (Q, W, C) is a 2-representation infinite algebra given in [4, Example 2.14], whose quiver is the left upper one in the picture below. We can consider the 2-APR tilting Λ-module T associated with vertex 2. By Theorem 7, $\Gamma := \mathrm{End}_\Lambda(T)$ is a 2-representation infinite algebra (this also follows from [4, Theorem 2.13]). Moreover T induces a tilting $\widehat{\Lambda}$-module $\widehat{T}$ with $\widehat{\Gamma} \simeq \mathrm{End}_{\widehat{\Lambda}}(\widehat{T}) \simeq \widehat{\Lambda}$.

In this example, the quiver of Γ can be calculated by the same argument of [5, Theorem 3.11], and the 3-preprojective alegbra $\widehat{\Lambda}$ is given as the Jacobian algebra of (Q, W).

Thus, we have the following diagram of quivers, where horizontal arrows indicate tilts of T and $\widehat{T}$, respectively, and vertical arrows indicate taking 3-preprojective algebras.



References

[1] C. Amiot, O. Iyama, I. Reiten, *Stable categories of Cohen-Macaulay modules and cluster categories*, Amer. J. Math. 137 (2015), no. 3, 813–857.

[2] A. B. Buan, O. Iyama, I. Reiten, J. Scott, *Cluster structures for 2-Calabi-Yau categories and unipotent groups*, Compos. Math. 145 (2009), 1035–1079.

[3] V. Ginzburg, *Calabi-Yau algebras*, arXiv:math/0612139.

[4] M. Herschend, O. Iyama, S. Oppermann, *n-representation infinite algebras*, Adv. Math. 252 (2014), 292–342.

[5] O. Iyama, S. Oppermann, *n-representation-finite algebras and n-APR tilting*, Trans. Amer. Math. Soc. 363 (2011), no. 12, 6575–6614.

[6] O. Iyama, S. Oppermann, *Stable categories of higher preprojective algebras*, Adv. Math. 244 (2013), 23–68.

[7] B. Keller, *Calabi-Yau triangulated categories*, Trends in representation theory of algebras and related topics, 467–489, EMS Ser. Congr. Rep., Eur. Math. Soc., Zurich, 2008.

[8] B. Keller, *Deformed Calabi-Yau completions. With an appendix by Michel Van den Bergh*, J. Reine Angew. Math. 654 (2011), 125–180.

[9] H. Minamoto, I. Mori, *The structure of AS-Gorenstein algebras*, Adv. Math. 226 (2011) 4061–4095.

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