TILTING OBJECTS FOR NONCOMMUTATIVE QUOTIENT SINGULARITIES

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Abstract. Tilting objects play a key role in the study of triangulated categories. Iyama and Takahashi proved that stable categories of graded maximal Cohen-Macaulay modules over Gorenstein isolated quotient singularities have tilting objects. As a consequence, it follows that these categories are triangle equivalent to derived categories of finite dimensional algebras. In this paper, using noncommutative algebraic geometry, we give a noncommutative generalization of Iyama and Takahashi's theorem with a more conceptual proof.

1. Introduction

In the study of triangulated categories, tilting objects play a key role. They often enable us to realize abstract triangulated categories as concrete derived categories of modules over algebras. One of the remarkable results on the existence of tilting objects has been obtained by Iyama and Takahashi.

Theorem 1. [2, Theorem 2.7, Corollary 2.10] *Let $S = k[x_1, \dots, x_d]$ be a polynomial algebra over an algebraically closed field k of characteristic 0 such that $\deg x_i = 1$ and $d \geq 2$. Let G be a finite subgroup of $\mathrm{SL}(d, k)$ acting linearly on S, and S^G the fixed subalgebra of S. Assume that S^G is an isolated singularity. Then the stable category $\underline{\mathsf{CM}}^{\mathbb{Z}}(S^G)$ of graded maximal Cohen-Macaulay modules has a tilting object. As a consequence, there exists a finite dimensional algebra Γ of finite global dimension such that*

$$\underline{\mathsf{CM}}^{\mathbb{Z}}(S^G) \cong \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\Gamma).$$

The stable categories of graded maximal Cohen-Macaulay modules are crucial objects studied in representation theory of algebras (see [1], [2] etc.) and also attract attention from the viewpoint of Kontsevich's homological mirror symmetry conjecture (see [3], [4] etc.). The aim of this paper is to generalize Theorem 1 to the noncommutative case using noncommutative algebraic geometry.

2. Noncommutative Gorenstein Isolated Quotient Singularities

In this section, we will explain how to consider a noncommutative version of a "Gorenstein isolated quotient singularity". Throughout this paper, we fix an algebraically closed field k. Unless otherwise stated, a graded algebra means an $\mathbb{N}$-graded algebra $A = \bigoplus_{i \in \mathbb{N}} A_i$ over k. We denote by $\mathsf{GrMod}\,A$ the category of graded right A-modules, and by $\mathsf{grmod}\,A$ the full subcategory consisting of finitely generated modules. Morphisms in $\mathsf{GrMod}\,A$ are right A-module homomorphisms of degree zero. Graded left A-modules are identified with

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graded A^o-modules where A^o is the opposite graded algebra of A. For $M \in \mathsf{GrMod}\, A$ and $n \in \mathbb{Z}$, we define $M_{\geq n} = \bigoplus_{i \geq n} M_i \in \mathsf{GrMod}\, A$, and $M(n) \in \mathsf{GrMod}\, A$ by $M(n) = M$ as an ungraded right A-module with the new grading $M(n)_i = M_{n+i}$. The rule $M \mapsto M(n)$ is a k-linear autoequivalence for $\mathsf{GrMod}\, A$ and $\mathsf{grmod}\, A$, called the shift functor. For $M, N \in \mathsf{GrMod}\, A$, we write the graded vector space

$$\underline{\mathrm{Ext}}^i_A(M, N) := \bigoplus_{n \in \mathbb{Z}} \mathrm{Ext}^i_{\mathsf{GrMod}\, A}(M, N(n)).$$

If $A_0 = k$, then we say that A is connected graded. Let A be a noetherian connected graded algebra. Then we view $k = A/A_{\geq 1} \in \mathsf{GrMod}\, A$ as a graded A-module.

Definition 2. A noetherian connected graded algebra A is called an AS-Gorenstein (resp. AS-regular) algebra of dimension d and of Gorenstein parameter ℓ if

- $\operatorname{injdim}_A A = \operatorname{injdim}_{A^o} A = d < \infty$ (resp. $\operatorname{gldim} A = d < \infty$), and
- $\underline{\mathrm{Ext}}^i_A(k, A) \cong \underline{\mathrm{Ext}}^i_{A^o}(k, A) \cong \begin{cases} k(\ell) & \text{if } i = d, \\ 0 & \text{if } i \neq d. \end{cases}$

Definition 3. Let A be a connected graded algebra. A linear resolution of $M \in \mathsf{GrMod}\, A$ is a minimal free resolution of the form

$$\cdots \to \bigoplus A(-i) \to \cdots \to \bigoplus A(-1) \to \bigoplus A \to M \to 0.$$

We say that A is Koszul if $k \in \mathsf{GrMod}\, A$ has a linear resolution.

It is well-known that if A is Koszul, then A is quadratic, and its dual graded algebra $A^!$ is also Koszul, which is called the Koszul dual of A.

Let S be an AS-regular Koszul algebra of dimension d. Then the Gorenstein parameter is $\ell = d$. It is known that S is commutative if and only if S is isomorphic to $k[x_1, \ldots, x_d]$ with $\deg x_i = 1$, so an AS-regular Koszul algebra is a noncommutative version of $k[x_1, \ldots, x_d]$ generated in degree 1.

Next we give some conventions on group actions on algebras used in this paper. Let A be a noetherian connected graded algebra. We denote by $\mathrm{GrAut}\, A$ the group of graded k-algebra automorphisms of A. Let $G \leq \mathrm{GrAut}\, A$ be a finite subgroup. Then the fixed subalgebra A^G and the skew group algebra $A * G$ are graded by $(A^G)_i = A^G \cap A_i$ and $(A * G)_i = A_i \otimes_k kG$ for $i \in \mathbb{N}$. We tacitly assume that $\operatorname{char} k$ does not divide $|G|$. Note that this condition is equivalent to the condition that kG is semi-simple. Two idempotent elements

$$e := \frac{1}{|G|} \sum_{g \in G} g, \quad \text{and} \quad e' := 1 - e$$

of kG play crucial roles in this study. Since $kG \subset A * G$, we often view e, e' as idempotent elements of $A * G$. It is well-known that the map $\varphi : A^G \to e(A * G)e$ defined by $\varphi(c) = e(c * 1)e$ is an isomorphism of graded algebras. Thus for any $M \in \mathsf{GrMod}\, A * G$, the right A^G-module structure on Me is given by identifying A^G with $e(A * G)e$ via φ.

In [5], the following characterization of isolated quotient singularities was given.

Proposition 4 ([5, Corollary 3.11])**.** *Let $S = k[x_1, \ldots, x_d]$ be a polynomial algebra generated in degree 1. If $\operatorname{char} k = 0$ and $G \leq \mathrm{SL}(d, k)$ is a finite subgroup, then the following are equivalent:*

(1) S^G *is an isolated singularity,*
(2) $S * G/(e)$ *is finite dimensional over* k.

Hence, combining the arguments of this section, if

- S is an AS-regular Koszul algebra of diminension d,
- $G \leq \mathrm{GrAut}\, S$ is a finite subgroup such that $\mathrm{char}\, k$ does not divide $|G|$,
- S^G is AS-Gorenstein, and
- $S * G/(e)$ is finite dimensional over k,

then S^G can be considered as a noncommutative Gorenstein isolated quotient singularity.

3. Main Results

Let A be an AS-Gorenstein algebra. Then $M \in \mathsf{grmod}\, A$ is called graded maximal Cohen-Macaulay if $\underline{\mathrm{Ext}}^i_A(M, A) = 0$ for all $i > 0$. We denote by $\mathsf{CM}^{\mathbb{Z}}(A)$ the full subcategory of $\mathsf{grmod}\, A$ consisting of graded maximal Cohen-Macaulay modules. Then $\mathsf{CM}^{\mathbb{Z}}(A)$ is a Frobenius category. The stable category of $\mathsf{CM}^{\mathbb{Z}}(A)$ is denoted by $\underline{\mathsf{CM}}^{\mathbb{Z}}(A)$. Note that $\underline{\mathsf{CM}}^{\mathbb{Z}}(A)$ is a triangulated category.

The following is the main result of this paper, saying that there exists a finite dimensional algebra Γ such that $\underline{\mathsf{CM}}^{\mathbb{Z}}(A) \cong \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, \Gamma)$ when A is a "noncommutative Gorenstein isolated quotient singularity".

Theorem 5 ([6])**.** *Let S be an AS-regular Koszul algebra of dimension $d \geq 2$, $G \leq \mathrm{GrAut}\, S$ a finite subgroup such that $\mathrm{char}\, k$ does not divide $|G|$, and let $e = \frac{1}{|G|}\sum_{g \in G} g \in kG \subset S * G$ and $e' = 1 - e$. Assume that S^G is AS-Gorenstein and $S * G/(e)$ is finite dimensional over k. If we define the graded right $S*G$-module U by $U = \bigoplus_{i=1}^d \Omega^i_{S*G} kG(i)$, then*

$$e'Ue$$

is a tilting object in $\underline{\mathsf{CM}}^{\mathbb{Z}}(S^G)$. As a consequence we have

$$\underline{\mathsf{CM}}^{\mathbb{Z}}(S^G) \cong \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\ \mathrm{End}_{\underline{\mathsf{CM}}^{\mathbb{Z}}(S^G)}(e'Ue)).$$

We remark that if S is commutative, then this theorem recovers Theorem 1.

For the rest of this section, we will explain how to calculate $\mathrm{End}_{\underline{\mathsf{CM}}^{\mathbb{Z}}(S^G)}(e'Ue)$. Let S be an AS-regular Koszul algebra of dimension $d \geq 2$, $G \leq \mathrm{GrAut}\, S$ a finite subgroup such that $\mathrm{char}\, k$ does not divide $|G|$, and let $e = \frac{1}{|G|}\sum_{g \in G} g \in kG \subset S * G$ and $e' = 1 - e$. Then we consider the Koszul dual algebra

$$S^! := \bigoplus_{i \in \mathbb{N}} \underline{\mathrm{Ext}}^i_S(S_0, S_0).$$

Since S is an AS-regular algebra, it is known that $S^!$ is a graded self-injective algebra (see [10]). Moreover, the opposite group G^o acts on $S^!$ as explained in [9]. We can define the ungraded finite dimensional algebra

$$\nabla(S^!) := \begin{pmatrix} S^!_0 & S^!_1 & \cdots & S^!_{d-1} \\ 0 & S^!_0 & \cdots & S^!_{d-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S^!_0 \end{pmatrix}$$

called the Beilinson algebra of $S^!$. Then we can also define the action of G^o on $\nabla(S^!)$. Thus we have the skew group algebra

$$\nabla(S^!) * G^o.$$

It is easy to check that $\nabla(S^!) * G^o$ has the idempotent

$$\tilde{e'} = \begin{pmatrix} e' & 0 & \cdots & 0 \\ 0 & e' & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e' \end{pmatrix} \in \begin{pmatrix} kG & * & \cdots & * \\ 0 & kG & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & kG \end{pmatrix} = \nabla(S^! * G^o) \cong \nabla(S^!) * G^o.$$

Theorem 6 ([6]). *As in the setting of Theorem 5, we have*

$$\operatorname{End}_{\underline{\mathsf{CM}}^{\mathbb{Z}}(S^G)}(e'Ue) \cong \tilde{e'}(\nabla(S^!) * G^o)\tilde{e'}$$

as algebras.

Thanks to this theorem, we can calculate the endomorphism algebra of the tilting object found in Theorem 5. In the next section, we present an example.

4. An Example

The aim of this section is to provide an explicit example of Theorem 5 and Theorem 6. In this section, we assume that k is an algebraically closed field of characteristic 0.

Example 7 ([6]). Let S be $k\langle x_1, x_2, x_3, x_4\rangle$ having six defining relations

$$x_1^2 + x_2^2, \quad x_1x_3 + x_3x_1, \quad x_1x_4 + x_4x_1, \quad x_2x_3 + x_3x_2, \quad x_2x_4 + x_4x_2, \quad x_3x_4 + x_4x_3,$$

with $\deg x_1 = \deg x_2 = \deg x_3 = \deg x_4 = 1$. Then S is a noetherian AS-regular Koszul algebra over k of dimension 4. Let G be a cyclic group generated by $g = \operatorname{diag}(1, -1, -1, -1)$. Then g defines a graded algebra automorphism of S, so G naturally acts on S. Clearly $|G| = 2$. One can check that S^G is AS-Gorenstein of dimension 4 (although $\det g \neq 1$). Moreover, by using a quiver presentation of $S * G/(e)$, we can check that $S * G/(e)$ is finite dimensional over k.

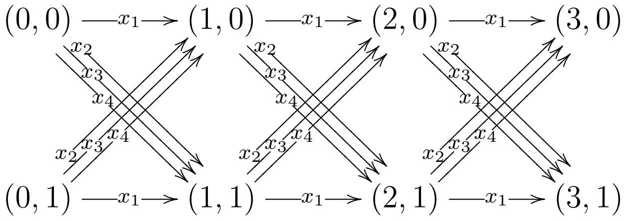
The Koszul dual $S^!$ is $k\langle x_1, x_2, x_3, x_4\rangle$ having ten defining relations

$$x_2^2 - x_1^2, \quad x_3x_1 - x_1x_3, \quad x_4x_1 - x_1x_4, \quad x_2x_1, \quad x_1x_2,$$
$$x_3x_2 - x_2x_3, \quad x_4x_2 - x_2x_4, \quad x_4x_3 - x_3x_4, \quad x_3^2, \quad x_4^2,$$

with $\deg x_1 = \deg x_2 = \deg x_3 = \deg x_4 = 1$. Then a quiver presentation of the Beilinson algebra $\nabla(S^!)$ is given as follows.

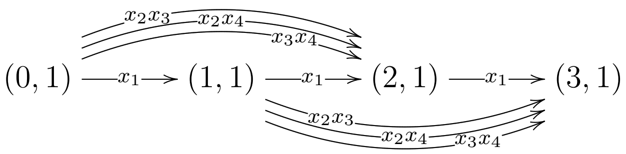
$$0 \begin{array}{c} \xrightarrow{x_1} \\ \xrightarrow{x_2} \\ \xrightarrow{x_3} \\ \xrightarrow{x_4} \end{array} 1 \begin{array}{c} \xrightarrow{x_1} \\ \xrightarrow{x_2} \\ \xrightarrow{x_3} \\ \xrightarrow{x_4} \end{array} 2 \begin{array}{c} \xrightarrow{x_1} \\ \xrightarrow{x_2} \\ \xrightarrow{x_3} \\ \xrightarrow{x_4} \end{array} 3 \qquad \begin{array}{l} x_1^2 = x_2^2,\ x_3x_1 = x_1x_3,\ x_4x_1 = x_1x_4, \\ x_3x_2 = x_2x_3,\ x_4x_2 = x_2x_4,\ x_4x_3 = x_3x_4, \\ x_2x_1 = x_1x_2 = x_3^2 = x_4^2 = 0. \end{array}$$

By [8, Section 2.3], it follows that a quiver presentation of the skew group algebra $\nabla(S^!) * G^o$ is



$$x_1^2 = x_2^2,\ x_3x_1 = x_1x_3,$$
$$x_1x_4 = x_4x_1,\ x_3x_2 = x_2x_3,$$
$$x_2x_4 = x_4x_2,\ x_4x_3 = x_3x_4,$$
$$x_2x_1 = x_1x_2 = x_3^2 = x_4^2 = 0.$$

Since $\tilde{e'} = (0,1) + (1,1) + (2,1) + (3,1)$ in the present setting, a quiver presentation of $\tilde{e'}(\nabla(S^!) * G^o)\tilde{e'}$ is obtained by



(4.1)

$$x_2x_3x_1 = x_1x_2x_3 = 0$$
$$x_2x_4x_1 = x_1x_2x_4 = 0$$
$$x_3x_4x_1 = x_1x_3x_4$$
$$x_1^3 = 0$$

Hence, if we denote by (Q, R) the quiver with relations in (4.1), then we have a triangle equivalence

$$\underline{\mathsf{CM}}^{\mathbb{Z}}(S^G) \cong \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, kQ/(R))$$

by Theorem 5 and Theorem 6.

5. Key Results for the Proof of Theorem 5

Our proof of Theorem 5 is different from Iyama and Takahashi's proof of Theorem 1. In this section, we summarize key results for our proof. We use techniques of noncommutative algebraic geometry.

Let A be a noetherian graded algebra. We denote by $\mathsf{tors}\, A$ the full subcategory of $\mathsf{grmod}\, A$ consisting of finite dimensional modules. The noncommutative projective scheme associated to A is defined by the quotient category

$$\mathsf{tails}\, A := \mathsf{grmod}\, A/\mathsf{tors}\, A.$$

If A is a commutative graded algebra finitely generated in degree 1 over k, then $\mathsf{tails}\, A$ is equivalent to the category of coherent sheaves on $\operatorname{Proj} A$ by results of Serre, justifying the terminology. We denote by $\pi : \mathsf{grmod}\, A \to \mathsf{tails}\, A$ the quotient functor.

Before proving Theorem 5, we prove the following result about tilting objects in the derived categories $\mathsf{D}^{\mathsf{b}}(\mathsf{tails}\, S^G)$.

Theorem 8 ([6])**.** *Let S be an AS-regular Koszul algebra of dimension $d \geq 2$, $G \leq \operatorname{GrAut} S$ a finite subgroup such that* $\operatorname{char} k$ *does not divide $|G|$, and let $e = \frac{1}{|G|}\sum_{g\in G} g \in kG \subset S * G$ and $e' = 1 - e$. Assume that $S * G/(e)$ is finite dimensional over k. If we consider the graded right $S * G$-module $U = \bigoplus_{i=1}^{d} \Omega^i_{S*G} kG(i)$, then*

$$\pi U e$$

is a tilting object in $\mathsf{D}^{\mathsf{b}}(\mathsf{tails}\, S^G)$.

We now give an outline of the proof of Theorem 8. First we show that $(S * G)^!$ is an $\mathbb{N}$-graded self-injective Koszul algebra. (See [6] for our definition of Koszul for (non-connected) $\mathbb{N}$-graded algebras.) Using the BGG correspondence and the isolated singularity property of S^G, we have

$$\mathsf{D}^{\mathsf{b}}(\mathsf{tails}\, S^G) \cong \underline{\mathsf{grmod}}(S * G)^!$$

as triangulated categories. Under this equivalence, we can show that πUe corresponds to the tilting object in $\underline{\mathsf{grmod}}(S*G)^!$ which was obtained by Yamaura [11, Theorem 3.3 (2)]. Thus Theorem 8 follows.

Furthermore, in the setting of Theorem 8, if S^G is AS-Gorenstein, then there exists an embedding

$$\underline{\mathsf{CM}}^{\mathbb{Z}}(S^G) \hookrightarrow \mathsf{D}^{\mathsf{b}}(\mathsf{tails}\, S^G)$$

by Orlov's theorem [7]. We can verify that $e'Ue$ is sent to $\pi e'Ue$.

Combining these results, we have

$$\begin{array}{ccccc} \underline{\mathsf{CM}}^{\mathbb{Z}}(S^G) & \hookrightarrow & \mathsf{D}^{\mathsf{b}}(\mathsf{tails}\, S^G) & \cong & \underline{\mathsf{grmod}}(S * G)^! \\ e'Ue & \mapsto & \pi e'Ue \quad \triangleleft\!\!\!\oplus \quad \pi Ue & \mapsto & Y \end{array}$$

where Y is the Yamaura tilting object. Using this, we can give a conceptual proof that $e'Ue$ is a tilting object in $\underline{\mathsf{CM}}^{\mathbb{Z}}(S^G)$ in terms of triangulated categories.

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