ON THE HOCHSCHILD COHOMOLOGY RING MODULO NILPOTENCE OF THE QUIVER ALGEBRA WITH QUANTUM-LIKE RELATIONS

DAIKI OBARA

Abstract. This paper is based on my talk given at the 48th Symposium on Ring Theory and Representation Theory held at Nagoya University, Japan, 7–10 September 2015. In this paper, we consider some example of finite dimensional quiver algebra over a field k with quantum-like relations. We determine the projective bimodule resolution of the algebras using the total complex. And we have the ring structure of the Hochschild cohomology ring modulo nilpotence.

Introduction

Let A be an indecomposable finite dimensional algebra over a field k and $\operatorname{char} k = 0$. We denote by A^e the enveloping algebra $A \otimes_k A^{op}$ of A, so that left A^e-modules correspond to A-bimodules. The Hochschild cohomology ring is given by $\mathrm{HH}^*(A) = \mathrm{Ext}^*_{A^e}(A, A) = \oplus_{n\geq 0}\mathrm{Ext}^n_{A^e}(A, A)$ with Yoneda product. It is well-known that $\mathrm{HH}^*(A)$ is a graded commutative ring, that is, for homogeneous elements $\eta \in \mathrm{HH}^m(A)$ and $\theta \in \mathrm{HH}^n(A)$, we have $\eta\theta = (-1)^{mn}\theta\eta$. Let $\mathcal{N}$ denote the ideal of $\mathrm{HH}^*(A)$ generated by all homogeneous nilpotent elements. Then $\mathcal{N}$ is contained in every maximal ideal of $\mathrm{HH}^*(A)$, so that the maximal ideals of $\mathrm{HH}^*(A)$ are in 1-1 correspondence with those in the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}$. In [6], Snashall and Solberg used the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}$ to define a support variety for any finitely generated module over A. This led us to consider the ring structure of $\mathrm{HH}^*(A)/\mathcal{N}$. In [5], Snashal gave the question whether we can give necessary and sufficient conditions on a finite dimensional algebra A for $\mathrm{HH}^*(A)/\mathcal{N}$ to be finitely generated as a k-algebra. With respect to sufficient condition, Green, Snashall and Solberg have shown that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated for self-injective algebras of finite representation type [1] and for monomial algebras [2].

Let Γ be the quiver with 4 vertices and 6 arrows as follows:

$$e_{(1,2)} \underset{a_{(1,2)}}{\overset{a_{(1,1)}}{\leftrightarrows}} e_{(1,1)} \underset{a_{(2,2)}}{\overset{a_{(2,1)}}{\rightleftarrows}} e_{(2,2)} \underset{a_{(3,2)}}{\overset{a_{(3,1)}}{\rightleftarrows}} e_{(3,2)}$$

and I the ideal of $k\Gamma$ generated by

$$X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, X_1X_2 - X_2X_1, X_2X_3 - X_3X_2,$$
$$a_{(1,2)}a_{(2,1)}X_2^{l}a_{(3,1)}, a_{(3,2)}a_{(2,2)}X_2^{l'}a_{(1,1)} \text{ for } 0 \leq l, l' \leq n_2 - 1.$$

The detailed version of this paper will be submitted for publication elsewhere.

where $X_i = (a_{i,1} + a_{i,2})^2$ and n_i are integers with $n_i \geq 2$ for $1 \leq i \leq 3$. Paths in Γ are written from left to right. In this paper, we consider the quiver algebra $A = k\Gamma/I$.

In this paper,we determine the projective bimodule resolution of this algebra A and the ring structure of the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}$.

In [3] and [4], we have the minimal projective bimodule resolution and the Hochschild cohomology ring modulo nilpotence of the quiver algebra defined by the two cycles and quantum-like relation. Then, the projective resolution of this algebra was given by the total complex depending on the projective bimodule resolutions of two Nakayama algebras. Similarly, the projective bimodule resolution of A is given by the total complex depending on the projective bimodule resolutions of the quiver algebra defined by two cycles and the Nakayama algebra, Using this resolution, we have the ring structure of the Hochschild cohomology ring of A modulo nilpotence.

The content of the paper is organized as follows. In Section 1, we give the complexes of the projective A-bimodule to form the total complex. In Section 2, we determine the projective bimodule resolution of A and the Hochschild cohomology ring of A modulo nilpotence.

1. The complexes of the projective A-bimodule

In this section, we we give the complexes to form the projective bimodule resolution of A. We set $e_{(1,1)} = e_{(2,1)}, e_{(2,2)} = e_{(3,1)}$. Let $\varepsilon_{(i,j,0),\{(s_1,s_2),(t_1,t_2)\}} = \varepsilon_{(i,j,0),\{(s_1,s_2),(t_1,t_2)\},(l_1,l_2)} = e_{(s_1,s_2)} \otimes e_{(t_1,t_2)}$. We define projective left A^e-modules, equivalently A-bimodules:

$$P_0 = A\varepsilon_{(0,0),\{(1,1),(1,1)\}}A \oplus A\varepsilon_{(0,0),\{(1,2),(1,2)\}}A \oplus A\varepsilon_{(0,0),\{(2,2),(2,2)\}}A \oplus A\varepsilon_{(0,0),\{(3,2),(3,2)\}}A,$$

$$Q_{(i,0,0)} = \begin{cases} \displaystyle\coprod_{k=1}^{2}(A\varepsilon_{(i,0,0),\{(k,1),(k,2)\}}A \oplus A\varepsilon_{(i,0,0),\{(k,2),(k,1)\}}A) \oplus \coprod_{\substack{l_1+l_2=i \\ l_1,l_2>0}} A\varepsilon_{(i,0,0),\{(1,1),(1,1)\},(l_1,l_2)}A & \text{if } i \text{ is odd,} \\ \displaystyle\coprod_{k=1}^{2} A\varepsilon_{(i,0,0),\{(k,2),(k,2)\}}A \oplus \coprod_{\substack{l_1+l_2=i \\ l_1,l_2\geq 0}} A\varepsilon_{(i,0,0),\{(1,1),(1,1)\},(l_1,l_2)}A & \text{if } i \text{ is even.} \end{cases}$$

$$Q_{(0,j,0)} = \begin{cases} A\varepsilon_{(0,j,0),\{(3,1),(3,2)\}}A \oplus A\varepsilon_{(0,j,0),\{(3,2),(3,1)\}}A & \text{if } j \text{ is odd,} \\ A\varepsilon_{(0,j,0),\{(3,1),(3,1)\}}A \oplus A\varepsilon_{(0,j,0),\{(3,2),(3,2)\}}A & \text{if } j \text{ is even.} \end{cases}$$

$$Q_{(i,j,0)} = \coprod_{\substack{l_1+l_2=i \\ l_1,l_2>0}} (A\varepsilon_{(i,j,0),\{(1,1),(3,1)\},(l_1,l_2)}A \oplus A\varepsilon_{(i,j,0),\{(3,1),(1,1)\},(l_1,l_2)}A) \oplus A\varepsilon_{(i,j,0),\{(3,1),(3,1)\}}A$$

$$\oplus \begin{cases} A\varepsilon_{(i,j,0),\{(1,1),(3,1)\}}A \oplus A\varepsilon_{(i,j,0),\{(3,1),(1,1)\}}A & \text{if } i,j \text{ are even,} \\ A\varepsilon_{(i,j,0),\{(1,1),(3,2)\}}A \oplus A\varepsilon_{(i,j,0),\{(3,2),(1,1)\}}A & \text{if } i \text{ is even and } j \text{ is odd,} \\ A\varepsilon_{(i,j,0),\{(1,2),(3,1)\}}A \oplus A\varepsilon_{(i,j,0),\{(3,1),(1,2)\}}A & \text{if } i \text{ is odd and } j \text{ is even,} \\ A\varepsilon_{(i,j,0),\{(1,2),(3,2)\}}A \oplus A\varepsilon_{(i,j,0),\{(3,2),(1,2)\}}A & \text{if } i,j \text{ are odd.} \end{cases}$$

$$
Q_{(i,1,k)} =
\begin{cases}
\left\{\begin{array}{l}
A\varepsilon_{(i,1,k),\{(1,1),(1,1)\},(1,i-1)}A \oplus A\varepsilon_{(i,1,k),\{(3,1),(3,1)\},(1,i-1)}A \\
\oplus \displaystyle\coprod_{\substack{l_1+l_2=i\\ l_1\geq 2,\, l_2\geq 1}} \coprod_{l'_1+l'_2=k+1} \Big(\coprod_{l'_1,l'_2:\text{even}} A\varepsilon_{(i,1,k),\{(1,1),(1,1)\},(l_1,l_2),(l'_1,l'_2)}A \\
\oplus \displaystyle\coprod_{l'_1,l'_2:\text{odd}} A\varepsilon_{(i,1,k),\{(3,1),(3,1)\},(l_1,l_2),(l'_1,l'_2)}A\Big) \oplus \begin{cases} A\varepsilon_{(i,1,1),\{(3,1),(3,1)\},(i,0)}A & \text{if } k=1,\\ 0 & \text{others,}\end{cases}
\end{array}\right. & \text{if } k \text{ is odd,}\\
\left\{\begin{array}{l}
A\varepsilon_{(i,1,k),\{(1,1),(3,1)\},(1,i-1)}A \oplus A\varepsilon_{(i,1,k),\{(3,1),(1,1)\},(1,i-1)}A \\
\oplus \displaystyle\coprod_{\substack{l_1+l_2=i\\ l_1\geq 2,\, l_2\geq 0}} \coprod_{l'_1+l'_2=k+1} \Big(\coprod_{l'_1:\text{odd},\,l'_2:\text{even}} A\varepsilon_{(i,1,k),\{(1,1),(3,1)\},(l_1,l_2),(l'_1,l'_2)}A \\
\oplus \displaystyle\coprod_{l'_1:\text{even},\,l'_2:\text{odd}} A\varepsilon_{(i,1,k),\{(3,1),(1,1)\},(l_1,l_2),(l'_1,l'_2)}A\Big)
\end{array}\right. & \text{if } k \text{ is even,}
\end{cases}
$$

$$
Q_{(i,j,k)} =
\begin{cases}
\left\{\begin{array}{l}
\displaystyle\coprod_{\substack{l_1+l_2=i\\ l_1\geq 2,\, l_2\geq 1}} A\varepsilon_{(i,j,k),\{(1,1),(1,1)\},(l_1,l_2),1}A \oplus A\varepsilon_{(i,j,k),\{(1,1),(1,1)\},(l_1,l_2),2}A \\
\displaystyle\coprod_{l'_1+l'_2=k+1} \Big(\coprod_{l'_1,l'_2:\text{odd}} A\varepsilon_{(i,j,k),\{(1,1),(1,1)\},(1,i-1),(l'_1,l'_2)}A \oplus \coprod_{l'_1,l'_2:\text{even}} A\varepsilon_{(i,j,k),\{(3,1),(3,1)\},(1,i-1),(l'_1,l'_2)}A\Big) \\
\oplus \begin{cases}
A\varepsilon_{(i,j,k),\{(1,2),(1,1)\},(i,0)}A \oplus A\varepsilon_{(i,j,k),\{(1,1),(1,2)\},(i,0)}A & \text{if } i \text{ is odd}, j \text{ is even},\\
A\varepsilon_{(i,j,k),\{(3,2),(3,1)\},(i,0)}A \oplus A\varepsilon_{(i,j,k),\{(3,1),(3,2)\},(i,0)}A & \text{if } i \text{ is even}, j \text{ is odd},\\
A\varepsilon_{(i,j,k),\{(1,1),(1,1)\},(i,0),1}A \oplus A\varepsilon_{(i,j,k),\{(1,1),(1,1)\},(i,0),2}A & \\
\quad \oplus A\varepsilon_{(i,j,k),\{(3,1),(3,1)\},(i,0),1}A \oplus A\varepsilon_{(i,j,k),\{(3,1),(3,1)\},(i,0),2}A & \text{if } i,j \text{ are even},\\
0 & \text{if } i,j \text{ are odd},
\end{cases}
\end{array}\right. & \text{if } k \text{ is odd,}\\
\left\{\begin{array}{l}
\displaystyle\coprod_{\substack{l_1+l_2=i\\ l_1\geq 2,\, l_2\geq 1}} A\varepsilon_{(i,j,k),\{(1,1),(3,1)\},(l_1,l_2)}A \oplus A\varepsilon_{(i,j,k),\{(3,1),(1,1)\},(l_1,l_2)}A \\
\displaystyle\coprod_{l'_1+l'_2=k+1} \Big(\coprod_{l'_1:\text{even},\,l'_2:\text{odd}} A\varepsilon_{(i,j,k),\{(1,1),(3,1)\},(1,i-1),(l'_1,l'_2)}A \oplus \coprod_{l'_1:\text{odd},\,l'_2:\text{even}} A\varepsilon_{(i,j,k),\{(3,1),(1,1)\},(1,i-1),(l'_1,l'_2)}A\Big) \\
\oplus \begin{cases}
A\varepsilon_{(i,j,k),\{(1,2),(3,1)\},(i,0)}A \oplus A\varepsilon_{(i,j,k),\{(3,1),(1,2)\},(i,0)}A & \text{if } i \text{ is odd}, j \text{ is even},\\
A\varepsilon_{(i,j,k),\{(3,2),(1,1)\},(i,0)}A \oplus A\varepsilon_{(i,j,k),\{(1,1),(3,2)\},(i,0)}A & \text{if } i \text{ is even}, j \text{ is odd},\\
A\varepsilon_{(i,j,k),\{(1,1),(3,1)\},(i,0),1}A \oplus A\varepsilon_{(i,j,k),\{(1,1),(3,1)\},(i,0),2}A & \\
\quad \oplus A\varepsilon_{(i,j,k),\{(3,1),(1,1)\},(i,0),1}A \oplus A\varepsilon_{(i,j,k),\{(3,1),(1,1)\},(i,0),2}A & \text{if } i,j \text{ are even},\\
0 & \text{if } i,j \text{ are odd},
\end{cases}
\end{array}\right. & \text{if } k \text{ is even,}
\end{cases}
$$

Let π: $P_0 \to A$ be the multiplication map. Then, we have the following complexes.

Proposition 1. *For $1 \leq k \leq 3$, we set*

$$
\begin{aligned}
E_{(i,j,0),(k,1)} &= \varepsilon_{(i,j,0),\{(k,1),(k,2)\}} a_{(k,2)} + a_{(k,1)} \varepsilon_{(i,j,0),\{(k,2),(k,1)\}},\\
E_{(i,j,0),(k,2)} &= \varepsilon_{(i,j,0),\{(k,2),(k,1)\}} a_{(k,1)} + a_{(k,2)} \varepsilon_{(i,j,0),\{(k,1),(k,2)\}}
\end{aligned}
$$

(1) *We have the following complex depending on the projective bimodule resolution of the quiver algebra defined by the two cycles and quantum-like relation.*

$$
P_0 \overset{\partial_{(1,0,0),1}}{\longleftarrow} Q_{(1,0,0)} \overset{\partial_{(2,0,0),1}}{\longleftarrow} \cdots \overset{\partial_{(n,0,0),1}}{\longleftarrow} Q_{(n,0,0)} \leftarrow \cdots .
$$

where, for $i \geq 0$, the left A^e-homomorphisms $\partial_{(i+1,0,0),1}$:$Q_{(i+1,0,0)} \to Q_{(i,0,0)}$ are defined as follows.

$$
\partial_{(i+1,0,0)} :
$$

$$
\begin{cases}
\begin{cases}
\varepsilon_{(i+1,0,0),\{(k,1),(k,2)\}} \mapsto \varepsilon_{(i,0,0),\{(k,1),(k,1)\},(i,0)} a_{(k,1)} - a_{(k,1)} \varepsilon_{(i,0,0),\{(k,2),(k,2)\}} & \text{for } 1 \leq k \leq 2,\\
\varepsilon_{(i+1,0,0),\{(k,2),(k,1)\}} \mapsto \varepsilon_{(i,0,0),\{(k,2),(k,2)\}} a_{(k,2)} - a_{(k,2)} \varepsilon_{(i,0,0),\{(k,1),(k,1)\},(i,0)} & \text{for } 1 \leq k \leq 2,\\
\varepsilon_{(i+1,0,0),\{(1,1),(1,1)\},(l_1,l_2)} \mapsto & \\
\begin{cases}
-\sum_{l=0}^{n_1-1} X_1^l \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(l_1-1,l_2)} X_1^{n_1-1-l} + \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(l_1,l_2-1)} X_2 & \\
\quad - X_2 \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(l_1,l_2-1)} & \text{if } l_1 \text{ is even and } l_2 \text{ is odd},\\
\varepsilon_{(i,0,0),\{(1,1),(1,1)\},(l_1-1,l_2)} X_1 - X_1 \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(l_1,l_2-1)} & \\
\quad + \sum_{l=0}^{n_2-1} X_2^l \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(l_1,l_2-1)} X_2^{n_2-1-l} & \text{if } l_1 \text{ is odd and } l_2 \text{ is even},
\end{cases}
\end{cases} & \text{if } i \text{ is odd},\\
\begin{cases}
\varepsilon_{(i+1,0,0),\{(1,1),(1,1)\},(i+1,0)} \mapsto \sum_{l=0}^{n_1-1} X_1^l E_{(i,0,0),(1,1)} X_1^{n_1-1-l}, & \\
\varepsilon_{(i+1,0,0),\{(2,1),(2,1)\},(0,i+1)} \mapsto \sum_{l=0}^{n_2-1} X_2^l E_{(i,0,0),(2,1)} X_2^{n_2-1-l}, & \\
\varepsilon_{(i+1,0,0),\{(k,2),(k,2)\}} \mapsto \sum_{l=0}^{n_k-1} X_k^l E_{(i,0,0),(k,2)} X_k^{n_k-1-l} \text{ for } 1 \leq k \leq 2, & \\
\varepsilon_{(i+1,0,0),\{(1,1),(1,1)\},(l_1,l_2)} \mapsto & \\
\begin{cases}
-E_{(1,0,0),(2,1)} X_1 + X_1 E_{(1,0,0),(2,1)} + E_{(1,0,0),(1,1)} X_2 - X_2 E_{(1,0,0),(1,1)} & \text{if } l_1 = 1, l_2 = 1,\\
-\varepsilon_{(i,0,0),\{(1,1),(1,1)\},(i-1,1)} X_1 + X_1 \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(i-1,1)} + E_{(i,0,0),(1,1)} X_2 - X_2 E_{(i,0,0),(1,1)} & \\
 & \text{if } l_1 = i, l_2 = 1,\\
-E_{(i,0,0),(2,1)} X_1 + X_1 E_{(i,0,0),(2,1)} + \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(1,i-1)} X_2 - X_2 \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(1,i-1)} & \\
 & \text{if } l_1 = 1, l_2 = i,\\
-\varepsilon_{(i,0,0),\{(1,1),(1,1)\},(l_1-1,l_2)} X_1 + X_1 \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(l_1-1,l_2)} & \\
\quad + \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(l_1,l_2-1)} X_2 - X_2 \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(l_1,l_2-1)} & \text{if } l_1, l_2 \text{ are odd},\\
\sum_{l=0}^{n_1-1} X_1^l \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(l_1-1,l_2)} X_1^{n_1-1-l} + \sum_{l=0}^{n_2-1} X_2^l \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(l_1,l_2-1)} X_2^{n_2-1-l} & \\
 & \text{if } l_1, l_2 \text{ are even},
\end{cases}
\end{cases} & \text{if } i \text{ is even},
\end{cases}
$$

(2) *We have the following complexe depending on the projective bimodule resolution of Nakayama algebra.*

$$P_0 \overset{\partial_{(0,1,0),2}}{\longleftarrow} Q_{(0,1,0)} \overset{\partial_{(0,2,0),2}}{\longleftarrow} \cdots \overset{\partial_{(0,n,0),2}}{\longleftarrow} Q_{(0,n,0)} \leftarrow \cdots .$$

where, for $j \geq 0$, the left A^e-homomorphisms $\partial_{(0,j+1,0),2}$:$Q_{(0,j+1,0)} \to Q_{(0,j,0)}$ are defined as follows.

$$\partial_{(0,j+1,0)}:$$
$$\begin{cases} \begin{cases} \varepsilon_{(0,j+1,0),\{(3,1),(3,2)\}} \mapsto \varepsilon_{(0,j,0),\{(3,1),(3,1)\}}a_{(3,1)} - a_{(3,1)}\varepsilon_{(0,j,0),\{(3,2),(3,2)\}}, \\ \varepsilon_{(0,j+1,0),\{(3,2),(3,1)\}} \mapsto \varepsilon_{(0,j,0),\{(3,2),(3,2)\}}a_{(3,2)} - a_{(3,2)}\varepsilon_{(0,j,0),\{(3,1),(3,1)\}}, \end{cases} & \textit{if } j \textit{ is even,} \\ \begin{cases} \varepsilon_{(0,j+1,0),\{(3,1),(3,1)\}} \mapsto \sum_{l=0}^{n_3-1} X_3^l \varepsilon_{(0,j,0),\{(3,1),(3,1)\}} X_3^{n_3-1-l}, \\ \varepsilon_{(0,j+1,0),\{(3,2),(3,2)\}} \mapsto \sum_{l=0}^{n_3-1} X_3^l \varepsilon_{(0,j,0),\{(3,2),(3,2)\}} X_3^{n_3-1-l} \end{cases} & \textit{if } j \textit{ is odd.} \end{cases}$$

(3) *We have the following complex depending on the relations $X_1^{n_1}$, $X_2^{n_2}$ and $X_1X_2 - X_2X_1$.*

$$Q_{(0,j,0)} \overset{\partial_{(1,j,0),1}}{\longleftarrow} Q_{(1,j,0)} \overset{\partial_{(2,j,0),1}}{\longleftarrow} \cdots \overset{\partial_{(n,j,0),1}}{\longleftarrow} Q_{(n,j,0)} \leftarrow \cdots .$$

where, for $i \geq 0$, the left A^e-homomorphisms $\partial_{(i,j,0),1}$:$Q_{(i,j,0)} \to Q_{(i-1,j,0)}$ are defined as follows.

$$\varepsilon_{(i,j,0),\{(1,2),(3,2)\}} \mapsto \begin{cases} a_{(1,2)}a_{(2,1)}\varepsilon_{(0,j,0),\{(3,1),(3,2)\}} & \textit{if } i = 1 \textit{ and } j \textit{ is odd,} \\ a_{(1,2)}\varepsilon_{(i-1,j,0),\{(1,1),(3,2)\}} & \textit{if } i \textit{ is odd}(\neq 1) \textit{ and } j \textit{ is odd,} \end{cases}$$
$$\varepsilon_{(i,j,0),\{(1,1),(3,2)\}} \mapsto X_1^{n_1-1}a_{(1,1)}\varepsilon_{(i-1,j,0),\{(1,2),(3,2)\}} \textit{ if } i \textit{ is even and } j \textit{ is odd,}$$
$$\varepsilon_{(i,j,0),\{(1,1),(3,1)\}} \mapsto X_1^{n_1-1}a_{(1,1)}\varepsilon_{(i-1,j,0),\{(1,2),(3,1)\}} \textit{ if } i, j \textit{ are even,}$$
$$\varepsilon_{(i,j,0),\{(3,2),(1,2)\}} \mapsto \begin{cases} \varepsilon_{(0,j,0),\{(3,2),(3,1)\}}a_{(2,2)}a_{(1,1)} & \textit{if } i = 1 \textit{ and } j \textit{ is odd,} \\ \varepsilon_{(i-1,j,0),\{(3,2),(1,1)\}}a_{(1,1)} & \textit{if } i \textit{ is odd}(\neq 1) \textit{ and } j \textit{ is odd,} \end{cases}$$
$$\varepsilon_{(i,j,0),\{(3,2),(1,1)\}} \mapsto \varepsilon_{(i-1,j,0),\{(3,2),(1,2)\}}a_{(1,2)}X_1^{n_1-1} \textit{if } i \textit{ is even and } j \textit{ is odd,}$$
$$\varepsilon_{(i,j,0),\{(3,1),(1,1)\}} \mapsto \varepsilon_{(i-1,j,0),\{(3,1),(1,2)\}}a_{(1,2)}X_1^{n_1-1} \textit{if } i, j \textit{ are even,}$$
$$\varepsilon_{(i,j,0),\{(1,2),(3,1)\}} \mapsto a_{(1,2)}\varepsilon_{(i-1,j,0),\{(1,1),(3,1)\}} \textit{ if } i \textit{ is odd and } j \textit{ is even,}$$
$$\varepsilon_{(i,j,0),\{(3,1),(1,2)\}} \mapsto \varepsilon_{(i-1,j,0),\{(3,1),(1,1)\}}a_{(1,1)}, \textit{ if } i \textit{ is odd and } j \textit{ is even,}$$
$$\varepsilon_{(i,j,0),\{(2,2),(2,2)\}} \mapsto$$
$$\begin{cases} E_{(0,j,0),(3,1)}X_2 - X_2E_{(0,j,0),(3,1)} & \textit{if } i = 1 \textit{ and } j \textit{ is odd,} \\ \varepsilon_{(i-1,j,0),\{(2,2),(2,2)\}}X_2 - X_2\varepsilon_{(0,j,0),\{(2,2),(2,2)\}} & \textit{if } i = 1 \textit{ and } j \textit{ is even or } i \textit{ is odd}(i \neq 1), \\ \sum_{l=0}^{n_2-1} X_2^l \varepsilon_{(i-1,j,0),\{(2,2),(2,2)\}} X_2^{n_2-1-l} & \textit{if } i \textit{ is even,} \end{cases}$$

$$\varepsilon_{(i,j,0),\{(1,1),(3,1)\},(l_1,l_2)} \mapsto$$

$$\begin{cases} \sum_{l=0}^{n_2-1} X_2^l \varepsilon_{(i-1,j,0),\{(1,1),(3,1)\},(l_1,l_2-1)} X_2^{n_2-1-l} + \begin{cases} X_1 a_{(2,1)} \varepsilon_{(i-1,j,0),\{(2,2),(2,2)\}} & \text{if } l_1 = 1, \\ X_1 \varepsilon_{(i-1,j,0),\{(1,1),(3,1)\},(l_1-1,l_2)} & \text{others,} \end{cases} \\ \qquad\qquad\qquad\qquad \text{if } l_1 \text{ is odd and } l_2 \text{ is even,} \\ X_1^{n_1-1} \varepsilon_{(i-1,j,0),\{(1,1),(3,1)\},(l_1-1,l_2)} \\ \quad - \begin{cases} \varepsilon_{(i-1,j,0),\{(1,1),(3,2)\}} a_{(3,2)} X_2 - X_2 \varepsilon_{(i-1,j,0),\{(1,1),(3,2)\}} a_{(3,2)} & \text{if } l_2 = 1, \\ \varepsilon_{(i-1,j,0),\{(1,1),(3,1)\},(l_1,l_2-1)} X_2 - X_2 \varepsilon_{(i-1,j,0),\{(1,1),(3,1)\},(l_1,l_2-1)} & \text{others,} \end{cases} \\ \qquad\qquad\qquad\qquad \text{if } l_1 \text{ is even and } l_2 \text{ is odd,} \\ \begin{cases} X_1 a_{(2,1)} \varepsilon_{(i-1,j,0),\{(2,2),(2,2)\}} & \text{if } l_1 = 1, \\ X_1 \varepsilon_{(i-1,j,0),\{(1,1),(3,1)\},(l_1-1,l_2)} & \text{others,} \end{cases} \\ \quad - \begin{cases} a_{(1,1)} \varepsilon_{(i-1,j,0),\{(1,2),(3,2)\}} a_{(3,2)} X_2 - X_2 a_{(1,1)} \varepsilon_{(i-1,j,0),\{(1,2),(3,2)\}} a_{(3,2)} & \text{if } l_2 = 1, \\ \varepsilon_{(i-1,j,0),\{(1,1),(3,1)\},(l_1,l_2-1)} X_2 - X_2 \varepsilon_{(i-1,j,0),\{(1,1),(3,1)\},(l_1,l_2-1)} & \text{others,} \end{cases} \\ \qquad\qquad\qquad\qquad \text{if } l_1, l_2 \text{ are odd,} \\ X_1^{n_1-1} \varepsilon_{(i-1,j,0),\{(1,1),(3,1)\},(l_1-1,l_2)} - \sum_{l=0}^{n_2-1} X_2^l \varepsilon_{(i-1,j,0),\{(1,1),(3,1)\},(l_1,l_2-1)} X_2^{n_2-1-l} \\ \qquad\qquad\qquad\qquad \text{if } l_1, l_2 \text{ are even,} \end{cases}$$

$$\varepsilon_{(i,j,0),\{(3,1),(1,1)\},(l_1,l_2)} \mapsto$$

$$\begin{cases} \sum_{l=0}^{n_2-1} X_2^l \varepsilon_{(i-1,j,0),\{(3,1),(1,1)\},(l_1,l_2-1)} X_2^{n_2-1-l} + \begin{cases} \varepsilon_{(i-1,j,0),\{(2,2),(2,2)\}} a_{(2,2)} X_1 & \text{if } l_1 = 1, \\ \varepsilon_{(i-1,j,0),\{(3,1),(1,1)\},(l_1-1,l_2)} X_1 & \text{others,} \end{cases} \\ \qquad\qquad\qquad\qquad \text{if } l_1 \text{ is odd and } l_2 \text{ is even,} \\ \varepsilon_{(i-1,j,0),\{(3,1),(1,1)\},(l_1-1,l_2)} X_1^{n_1-1} \\ \quad - \begin{cases} a_{(3,1)} \varepsilon_{(i-1,j,0),\{(3,2),(1,1)\}} X_2 - X_2 a_{(3,1)} \varepsilon_{(i-1,j,0),\{(3,2),(1,1)\}} & \text{if } l_2 = 1, \\ \varepsilon_{(i-1,j,0),\{(3,1),(1,1)\},(l_1,l_2-1)} X_2 - X_2 \varepsilon_{(i-1,j,0),\{(3,1),(1,1)\},(l_1,l_2-1)} & \text{others,} \end{cases} \\ \qquad\qquad\qquad\qquad \text{if } l_1 \text{ is even and } l_2 \text{ is odd,} \\ \begin{cases} \varepsilon_{(i-1,j,0),\{(2,2),(2,2)\}} a_{(2,2)} X_1 & \text{if } l_1 = 1, \\ \varepsilon_{(i-1,j,0),\{(3,1),(1,1)\},(l_1-1,l_2)} X_1 & \text{others,} \end{cases} \\ \quad - \begin{cases} a_{(3,1)} \varepsilon_{(i-1,j,0),\{(3,2),(1,2)\}} a_{(1,2)} X_2 - X_2 a_{(3,1)} \varepsilon_{(i-1,j,0),\{(3,2),(1,2)\}} a_{(1,2)} & \text{if } l_2 = 1, \\ \varepsilon_{(i-1,j,0),\{(3,1),(1,1)\},(l_1,l_2-1)} X_2 - X_2 \varepsilon_{(i-1,j,0),\{(3,1),(1,1)\},(l_1,l_2-1)} & \text{others,} \end{cases} \\ \qquad\qquad\qquad\qquad \text{if } l_1, l_2 \text{ are odd,} \\ \varepsilon_{(i-1,j,0),\{(3,1),(1,1)\},(l_1-1,l_2)} X_1^{n_1-1} - \sum_{l=0}^{n_2-1} X_2^l \varepsilon_{(i-1,j,0),\{(3,1),(1,1)\},(l_1,l_2-1)} X_2^{n_2-1-l} \\ \qquad\qquad\qquad\qquad \text{if } l_1, l_2 \text{ are even,} \end{cases}$$

(4) *We have the following complex depending on the relation* $X_3^{n_3}$.

$$Q_{(i,0,0)} \overset{\partial_{(i,1,0),2}}{\longleftarrow} Q_{(i,1,0)} \overset{\partial_{(i,2,0),2}}{\longleftarrow} \cdots \overset{\partial_{(i,n,0),2}}{\longleftarrow} Q_{(i,n,0)} \leftarrow \cdots .$$

where, for $j \geq 0$*, the left* A^e*-homomorphisms* $\partial_{(i,j,0),2}: Q_{(i,j,0)} \to Q_{(i,j-1,0)}$ *are defined as follows.*

$$\varepsilon_{(i,j,0),\{(1,2),(3,2)\}} \mapsto$$
$$\begin{cases} \varepsilon_{(1,0,0),\{(1,2),(1,1)\}} a_{(2,1)} a_{(3,1)} + a_{(1,2)} \varepsilon_{(1,0,0),\{(2,1),(2,2)\}} a_{(3,1)} & \text{if } i = j = 1, \\ \varepsilon_{(i,0,0),\{(1,2),(1,1)\}} a_{(2,1)} a_{(3,1)} & \text{if } i \text{ is odd}(\neq 1) \text{ and } j = 1, \\ \varepsilon_{(i,j-1,0),\{(1,2),(3,1)\},(l_1,l_2)} a_{(3,1)} & \text{if } i, j \text{ is odd}(\neq 1), \end{cases}$$

$$\varepsilon_{(i,j,0),\{(3,2),(1,2)\}} \mapsto$$
$$\begin{cases} a_{(3,2)} a_{(2,2)} \varepsilon_{(1,0,0),\{(1,1),(1,2)\}} + a_{(3,2)} \varepsilon_{(1,0,0),\{(2,2),(2,1)\}} a_{(1,1)} & \text{if } i = j = 1, \\ a_{(3,2)} a_{(2,2)} \varepsilon_{(i,0,0),\{(1,1),(1,2)\}} & \text{if } i \text{ is odd}(\neq 1) \text{ and } j = 1, \\ a_{(3,2)} \varepsilon_{(i,j-1,0),\{(3,1),(1,2)\},(l_1,l_2)} & \text{if } i, j \text{ is odd}(\neq 1), \end{cases}$$

$$\varepsilon_{(i,j,0),\{(1,1),(3,2)\}} \mapsto \begin{cases} \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(i,0)} a_{(2,1)} a_{(3,1)} & \text{if } i \text{ is even and } j = 1, \\ \varepsilon_{(i,j-1,0),\{(1,1),(3,1)\},(l_1,l_2)} a_{(3,1)} & \text{if } i \text{ is even and } j \text{ is odd}(\neq 1), \end{cases}$$

$$\varepsilon_{(i,j,0),\{(3,2),(1,1)\}} \mapsto \begin{cases} a_{(3,2)} a_{(2,2)} \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(i,0)} & \text{if } i \text{ is even and } j = 1, \\ a_{(3,2)} \varepsilon_{(i,j-1,0),\{(3,1),(1,1)\},(l_1,l_2)} & \text{if } i \text{ is even and } j \text{ is odd}(\neq 1), \end{cases}$$

$$\varepsilon_{(i,j,0),\{(1,2),(3,1)\}} \mapsto \varepsilon_{(i,j-1,0),\{(1,2),(3,2)\},(l_1,l_2)} a_{(3,2)} X_3^{n_3-1} \quad \text{if } i \text{ is odd and } j \text{ is even},$$
$$\varepsilon_{(i,j,0),\{(3,1),(1,2)\}} \mapsto X_3^{n_3-1} a_{(3,1)} \varepsilon_{(i,j-1,0),\{(3,2),(1,2)\},(l_1,l_2)} \quad \text{if } i \text{ is odd and } j \text{ is even},$$
$$\varepsilon_{(i,j,0),\{(1,1),(3,1)\}} \mapsto \varepsilon_{(i,j-1,0),\{(1,1),(3,2)\},(l_1,l_2)} a_{(3,2)} X_3^{n_3-1} \quad \text{if } i, j \text{ are even},$$
$$\varepsilon_{(i,j,0),\{(3,1),(1,1)\}} \mapsto X_3^{n_3-1} a_{(3,1)} \varepsilon_{(i,j-1,0),\{(3,2),(1,1)\},(l_1,l_2)} \quad \text{if } i, j \text{ are even},$$

$$\varepsilon_{(i,j,0),\{(1,1),(3,1)\},(l_1,l_2)} \mapsto$$
$$\begin{cases} \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(1.i-1)} a_{(2,1)} X_3 + X_1 \varepsilon_{(i,0,0),\{(2,1),(2,2)\}} X_3 & \text{if } i \text{ is odd}(\neq 1), j = 1 \text{ and } l_1 = 1, \\ \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(l_1,l_2)} a_{(2,1)} X_3 & \text{if } j = 1 \text{ and ``} l_1 \neq 1 \text{ or } i \text{ is even''}, \\ \varepsilon_{(i,j-1,0),\{(1,1),(3,1)\},(l_1,l_2)} X_3 & \text{if } j \text{ is odd}(\neq 1), \\ a_{(1,1)} \varepsilon_{(i,j-1,0),\{(1,2),(3,2)\},(1,i-1)} a_{(3,2)} X_3^{n_3-1} & \text{if } i \text{ is odd}(\neq 1), j = 2 \text{ and } l_1 = 1, \\ \varepsilon_{(i,j-1,0),\{(1,1),(3,1)\},(l_1,l_2)} X_3^{n_3-1} & \text{others}, \end{cases}$$

$$\varepsilon_{(i,j,0),\{(3,1),(1,1)\},(l_1,l_2)} \mapsto$$
$$\begin{cases} X_3 a_{(2,2)} \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(1.i-1)} + X_3 \varepsilon_{(i,0,0),\{(2,2),(2,1)\}} X_1 & \text{if } i \text{ is odd}(\neq 1), j = 1 \text{ and } l_1 = 1, \\ X_3 a_{(2,2)} \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(l_1,l_2)} & \text{if } j = 1 \text{ and ``} l_1 \neq 1 \text{ or } i \text{ is even''}, \\ X_3 \varepsilon_{(i,j-1,0),\{(3,1),(1,1)\},(l_1,l_2)} & \text{if } j \text{ is odd}(\neq 1), \\ X_3^{n_3-1} a_{(3,1)} \varepsilon_{(i,j-1,0),\{(3,2),(1,2)\},(1,i-1)} a_{(1,2)} & \text{if } i \text{ is odd}(\neq 1), j = 2 \text{ and } l_1 = 1, \\ X_3^{n_3-1} \varepsilon_{(i,j-1,0),\{(3,1),(1,1)\},(l_1,l_2)} & \text{others}, \end{cases}$$

$$\varepsilon_{(i,j,0),\{(3,1),(3,1)\}} \mapsto \begin{cases} \sum_{l=0}^{n_3-1} X_3^l \varepsilon_{(i,j-1,0),\{(3,1),(3,1)\}} X_3^{n_3-1-l} & \text{if } j \text{ is even}, \\ E_{(i,0,0),(2,2)} X_3 - X_3 E_{(i,0,0),(2,2)} & \text{if } i \text{ is odd and } j = 1, \\ \varepsilon_{(i,j-1,0),\{(3,1),(3,1)\}} X_3 - X_3 \varepsilon_{(i,j-1,0),\{(3,1),(3,1)\}} & \text{others}. \end{cases}$$

(5) *We have the following complexes depending on the relations* $a_{1,2}a_{(2,1)}X_2^la_{(3,1)}$ *and* $a_{3,2}a_{(2,2)}X_2^{l'}a_{(1,1)}$.

$$Q_{(i,j,0)} \overset{\partial_{(i,j,1),3}}{\longleftarrow} Q_{(i,j,1)} \overset{\partial_{(i,j,2),3}}{\longleftarrow} \cdots \overset{\partial_{(i,j,n),3}}{\longleftarrow} Q_{(i,j,n)} \leftarrow \cdots .$$

for $i \geq 2$, $j \geq 2$ *where, for* $k \geq 0$, *the left* A^e*-homomorphisms* $\partial_{(i,1,k),3}{:}Q_{(i,1,k)} \to Q_{(i,1,k-1)}$ *and* $\partial_{(i,j,k),3}{:}Q_{(i,j,k)} \to Q_{(i,j,k-1)}$ *are defined as follows.*

$$\begin{cases} \varepsilon_{(i,1,k),\{(1,1),(1,1)\},(1,i-1)} \mapsto \varepsilon_{(i,1,k-1),\{(1,1),(3,1)\},(1,i-1)}a_{(2,2)}X_1 + X_1a_{(2,1)}\varepsilon_{(i,1,k-1),\{(3,1),(1,1)\},(1,i-1)}, \\ \varepsilon_{(i,1,k),\{(3,1),(3,1)\},(1,i-1)} \mapsto \varepsilon_{(i,1,k-1),\{(3,1),(1,1)\},(1,i-1)}a_{(2,1)}X_3 - X_3a_{(2,2)}\varepsilon_{(i,1,k-1),\{(1,1),(3,1)\},(1,i-1)}, \\ \varepsilon_{(i,1,k),\{(1,1),(1,1)\},(l_1,l_2),(l'_1,l'_2)} \mapsto \\ \varepsilon_{(i,1,k-1),\{(1,1),(3,1)\},(l_1,l_2),(l'_1-1,l'_2)}a_{(2,2)}X_1 + X_1a_{(2,1)}\varepsilon_{(i,1,k-1),\{(3,1),(1,1)\},(l_1,l_2),(l'_1,l'_2-1)}, \\ \varepsilon_{(i,1,k),\{(3,1),(3,1)\},(l_1,l_2),(l'_1,l'_2)} \mapsto \\ \varepsilon_{(i,1,k-1),\{(3,1),(1,1)\},(l_1,l_2),(l'_1-1,l'_2)}a_{(2,1)}X_3 - X_3a_{(2,2)}\varepsilon_{(i,1,k-1),\{(1,1),(3,1)\},(l_1,l_2),(l'_1,l'_2-1)}, \\ \varepsilon_{(i,1,1),\{(3,1),(3,1)\},(i,0)} \mapsto a_{(3,1)}\varepsilon_{(i,1,0),\{(3,2),(1,1)\},(i,0)}a_{(2,1)}X_3 - X_3a_{(2,2)}\varepsilon_{(i,1,0),\{(1,1),(3,2)\},(i,0)}a_{(3,2)}, \end{cases}$$

if $j = 1$ *and* k *is odd,*

$$\begin{cases} \varepsilon_{(i,1,k),\{(1,1),(3,1)\},(1,i-1)} \mapsto \varepsilon_{(i,1,k-1),\{(1,1),(1,1)\},(1,i-1)}a_{(2,1)}X_3 - X_1a_{(2,1)}\varepsilon_{(i,1,k-1),\{(3,1),(3,1)\},(1,i-1)}, \\ \varepsilon_{(i,1,k),\{(3,1),(1,1)\},(1,i-1)} \mapsto \varepsilon_{(i,1,k-1),\{(3,1),(3,1)\},(1,i-1)}a_{(2,2)}X_1 - X_3a_{(2,2)}\varepsilon_{(i,1,k-1),\{(1,1),(1,1)\},(1,i-1)}, \\ \varepsilon_{(i,1,k),\{(1,1),(3,1)\},(l_1,l_2),(l'_1,l'_2)} \mapsto \\ \varepsilon_{(i,1,k-1),\{(1,1),(1,1)\},(l_1,l_2),(l'_1-1,l'_2)}a_{(2,1)}X_3 - X_1a_{(2,1)}\varepsilon_{(i,1,k-1),\{(3,1),(3,1)\},(l_1,l_2),(l'_1,l'_2-1)}, \\ \varepsilon_{(i,1,k),\{(3,1),(1,1)\},(l_1,l_2),(l'_1,l'_2)} \mapsto \\ \varepsilon_{(i,1,k-1),\{(3,1),(3,1)\},(l_1,l_2),(l'_1-1,l'_2)}a_{(2,2)}X_1 + X_3a_{(2,2)}\varepsilon_{(i,1,k-1),\{(1,1),(1,1)\},(l_1,l_2),(l'_1,l'_2-1)}, \end{cases}$$

if $j = 1$ *and* k *is even,*

$$\begin{cases} \varepsilon_{(i,j,k),\{(1,1),(1,1)\},(1,i-1),(l'_1,l'_2)} \mapsto \\ \varepsilon_{(i,j,k-1),\{(1,1),(3,1)\},(1,i-1),(l'_1-1,l'_2)}a_{(2,2)}X_1 - X_1a_{(2,1)}\varepsilon_{(i,j,k-1),\{(3,1),(1,1)\},(1,i-1),(l'_1,l'_2-1)}, \\ \varepsilon_{(i,j,k),\{(3,1),(3,1)\},(1,i-1),(l'_1,l'_2)} \mapsto \\ \varepsilon_{(i,j,k-1),\{(3,1),(1,1)\},(1,i-1),(l'_1-1,l'_2)}a_{(2,1)}X_3 + X_3a_{(2,2)}\varepsilon_{(i,j,k-1),\{(1,1),(3,1)\},(1,i-1),(l'_1,l'_2-1)}, \\ \varepsilon_{(i,j,k),\{(1,1),(1,1)\},(l_1,l_2),1} \mapsto \varepsilon_{(i,j,k-1),\{(1,1),(3,1)\},(l_1,l_2)}a_{(2,2)}X_1, \\ \varepsilon_{(i,j,k),\{(1,1),(1,1)\},(l_1,l_2),2} \mapsto X_1a_{(2,1)}\varepsilon_{(i,j,k-1),\{(3,1),(1,1)\},(l_1,l_2)}, \\ \varepsilon_{(i,j,k),\{(1,2),(1,1)\},(i,0)} \mapsto \varepsilon_{(i,j,k-1),\{(1,2),(3,1)\},(i,0)}a_{(2,2)}X_1, \\ \varepsilon_{(i,j,k),\{(1,1),(1,2)\},(i,0)} \mapsto X_1a_{(2,1)}\varepsilon_{(i,j,k-1),\{(3,1),(1,2)\},(i,0)}, \\ \varepsilon_{(i,j,k),\{(1,1),(1,1)\},(i,0),1} \mapsto \varepsilon_{(i,j,k-1),\{(1,1),(3,1)\},(i,0),1}a_{(2,2)}X_1, \\ \varepsilon_{(i,j,k),\{(1,1),(1,1)\},(i,0),2} \mapsto X_1a_{(2,1)}\varepsilon_{(i,j,k-1),\{(3,1),(1,1)\},(i,0),1}, \\ \varepsilon_{(i,j,k),\{(3,1),(3,1)\},(i,0),1} \mapsto \varepsilon_{(i,j,k-1),\{(3,1),(1,1)\},(i,0),2}a_{(2,1)}X_3, \\ \varepsilon_{(i,j,k),\{(3,1),(3,1)\},(i,0),2} \mapsto X_3a_{(2,2)}\varepsilon_{(i,j,k-1),\{(1,1),(3,1)\},(i,0),2}, \\ \varepsilon_{(i,j,k),\{(3,2),(3,1)\},(i,0)} \mapsto \varepsilon_{(i,j,k-1),\{(3,2),(1,1)\},(i,0)}a_{(2,1)}X_3, \\ \varepsilon_{(i,j,k),\{(3,1),(3,2)\},(i,0)} \mapsto X_3a_{(2,2)}\varepsilon_{(i,j,k-1),\{(1,1),(3,2)\},(i,0)}, \end{cases}$$

if $j \neq 1$ *and* k *is odd,*

$$\begin{cases}\varepsilon_{(i,j,k),\{(1,1),(3,1)\},(1,i-1),(l_1',l_2')} \mapsto \\ \varepsilon_{(i,j,k-1),\{(1,1),(1,1)\},(1,i-1),(l_1'-1,l_2')}a_{(2,1)}X_3 + X_1a_{(2,1)}\varepsilon_{(i,j,k-1),\{(3,1),(3,1)\},(1,i-1),(l_1',l_2'-1)}, \\ \varepsilon_{(i,j,k),\{(3,1),(1,1)\},(1,i-1),(l_1',l_2')} \mapsto \\ \varepsilon_{(i,j,k-1),\{(3,1),(3,1)\},(1,i-1),(l_1'-1,l_2')}a_{(2,2)}X_1 - X_3a_{(2,2)}\varepsilon_{(i,j,k-1),\{(1,1),(1,1)\},(1,i-1),(l_1',l_2'-1)}, \\ \varepsilon_{(i,j,k),\{(1,1),(3,1)\},(l_1,l_2)} \mapsto \varepsilon_{(i,j,k-1),\{(1,1),(1,1)\},(l_1,l_2),1}a_{(2,1)}X_3, \\ \varepsilon_{(i,j,k),\{(3,1),(1,1)\},(l_1,l_2)} \mapsto X_3a_{(2,2)}\varepsilon_{(i,j,k-1),\{(1,1),(1,1)\},(l_1,l_2),2}, \\ \varepsilon_{(i,j,k),\{(1,2),(3,1)\},(i,0)} \mapsto \varepsilon_{(i,j,k-1),\{(1,2),(1,1)\},(i,0)}a_{(2,1)}X_3, \\ \varepsilon_{(i,j,k),\{(3,1),(1,2)\},(i,0)} \mapsto X_3a_{(2,2)}\varepsilon_{(i,j,k-1),\{(1,1),(1,2)\},(i,0)}, \\ \varepsilon_{(i,j,k),\{(1,1),(3,1)\},(i,0),1} \mapsto \varepsilon_{(i,j,k-1),\{(1,1),(1,1)\},(i,0),1}a_{(2,1)}X_3, \\ \varepsilon_{(i,j,k),\{(3,1),(1,1)\},(i,0),1} \mapsto X_3a_{(2,2)}\varepsilon_{(i,j,k-1),\{(1,1),(1,1)\},(i,0),2}, \\ \varepsilon_{(i,j,k),\{(3,1),(1,1)\},(i,0),2} \mapsto \varepsilon_{(i,j,k-1),\{(3,1),(3,1)\},(i,0),1}a_{(2,2)}X_1, \\ \varepsilon_{(i,j,k),\{(1,1),(3,1)\},(i,0),2} \mapsto X_1a_{(2,1)}\varepsilon_{(i,j,k-1),\{(3,1),(3,1)\},(i,0),2}, \\ \varepsilon_{(i,j,k),\{(3,2),(1,1)\},(i,0)} \mapsto \varepsilon_{(i,j,k-1),\{(3,2),(3,1)\},(i,0)}a_{(2,2)}X_1, \\ \varepsilon_{(i,j,k),\{(1,1),(3,2)\},(i,0),2} \mapsto X_1a_{(2,1)}\varepsilon_{(i,j,k-1),\{(3,1),(3,2)\},(i,0)},\end{cases}$$

if $j \neq 1$ and k is even.

(6) *The following complex depend on the complexes of* (3):

$$0 \leftarrow Q_{(1,j,k)} \overset{\partial_{(2,j,k),1}}{\longleftarrow} Q_{(2,j,k)} \overset{\partial_{(3,j,k),1}}{\longleftarrow} \cdots \overset{\partial_{(n,j,k),1}}{\longleftarrow} Q_{(n,j,k)} \leftarrow \cdots .$$

The left A^e-homomorphism $\partial_{(i,j,k),1}$:$Q_{(i,j,k)} \to Q_{(i-1,j,k)}$ is defined as follows. We consider the case that k is odd. In the case k is even, we have the similar result.

$$\varepsilon_{(i,1,k),\{(1,1),(1,1)\},(1,i-1)} \mapsto$$
$$\begin{cases}0 & \text{if } i=2, \\ -\varepsilon_{(i-1,1,k),\{(1,1),(1,1)\},(1,i-2)}X_2 + X_2\varepsilon_{(i-1,1,k),\{(1,1),(1,1)\},(1,i-2)} & \text{if } i \text{ is even}(\neq 2), \\ \sum_{l=0}^{n_2-1} X_2^l\varepsilon_{(i-1,1,k),\{(1,1),(1,1)\},(1,i-2)}X_2^{n_2-1-l} & \text{if } i \text{ is odd,}\end{cases}$$

$$\varepsilon_{(i,1,k),\{(3,1),(3,1)\},(1,i-1)} \mapsto$$
$$\begin{cases}0 & \text{if } i=2, \\ -\varepsilon_{(i-1,1,k),\{(3,1),(3,1)\},(1,i-2)}X_2 + X_2\varepsilon_{(i-1,1,k),\{(3,1),(3,1)\},(1,i-2)} & \text{if } i \text{ is even}(\neq 2), \\ \sum_{l=0}^{n_2-1} X_2^l\varepsilon_{(i-1,1,k),\{(3,1),(3,1)\},(1,i-2)}X_2^{n_2-1-l} & \text{if } i \text{ is odd,}\end{cases}$$

$$\varepsilon_{(i,1,k),\{(1,1),(1,1)\},(l_1,l_2),(l_1',l_2')} \mapsto$$
$$\begin{cases}-\sum_{l=0}^{n_2-1} X_2^l\varepsilon_{(i-1,1,k),\{(1,1),(1,1)\},(l_1,l_2-1),(l_1',l_2')}X_2^{n_2-1-l} & \text{if } l_2 \text{ is even,} \\ -\varepsilon_{(i-1,1,k),\{(1,1),(1,1)\},(l_1,l_2-1),(l_1',l_2')}X_2 + X_2\varepsilon_{(i-1,1,k),\{(1,1),(1,1)\},(l_1,l_2-1),(l_1',l_2')} & \text{if } l_2 \text{ is odd,}\end{cases}$$
$$+\begin{cases}X_1^{n_1-1}\varepsilon_{(i-1,1,k),\{(1,1),(1,1)\},(l_1-1,l_2),(k+1,0)} & \text{if } l_1 \text{ is even and } l_1'=k+1, \\ X_1\varepsilon_{(i-1,1,k),\{(1,1),(1,1)\},(l_1-1,l_2),(k+1,0)} & \text{if } l_1 \text{ is odd and } l_1'=k+1, \\ \varepsilon_{(i-1,1,k),\{(1,1),(1,1)\},(l_1-1,l_2),(0,k+1)}X_1^{n_1-1} & \text{if } l_1 \text{ is even and } l_1'=0, \\ \varepsilon_{(i-1,1,k),\{(1,1),(1,1)\},(l_1-1,l_2),(0,k+1)}X_1 & \text{if } l_1 \text{ is odd and } l_1'=0, \\ 0 & \text{others,}\end{cases}$$

$$\varepsilon_{(i,1,k),\{(3,1),(3,1)\},(l_1,l_2),(l'_1,l'_2)} \mapsto$$
$$\begin{cases} -\sum_{l=0}^{n_2-1} X_2^l \varepsilon_{(i-1,1,k),\{(3,1),(3,1)\},(l_1,l_2-1),(l'_1,l'_2)} X_2^{n_2-1-l} & \textit{if } l_2 \textit{ is even}, \\ -\varepsilon_{(i-1,1,k),\{(3,1),(3,1)\},(l_1,l_2-1),(l'_1,l'_2)} X_2 + X_2 \varepsilon_{(i-1,1,k),\{(3,1),(3,1)\},(l_1,l_2-1),(l'_1,l'_2)} & \textit{if } l_2 \textit{ is odd}, \end{cases}$$
$$\varepsilon_{(i,1,1),\{(3,1),(3,1)\},(i,0)} \mapsto 0 \textit{ if } k = 1.$$

$$\varepsilon_{(i,j,k),\{(1,1),(1,1)\},(1,i-1),(l'_1,l'_2)} \mapsto$$
$$\begin{cases} 0 & \textit{if } i = 2, \\ -\varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(1,i-2),(l'_1,l'_2)} X_2 + X_2 \varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(1,i-2),(l'_1,l'_2)} & \textit{if } i \textit{ is even}(\neq 2), \\ \sum_{l=0}^{n_2-1} X_2^l \varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(1,i-2),(l'_1,l'_2)} X_2^{n_2-1-l} & \textit{if } i \textit{ is odd}, \end{cases}$$
$$\varepsilon_{(i,j,k),\{(3,1),(3,1)\},(1,i-1),(l'_1,l'_2)} \mapsto$$
$$\begin{cases} 0 & \textit{if } i = 2, \\ -\varepsilon_{(i-1,j,k),\{(3,1),(3,1)\},(1,i-2),(l'_1,l'_2)} X_2 + X_2 \varepsilon_{(i-1,j,k),\{(3,1),(3,1)\},(1,i-2),(l'_1,l'_2)} & \textit{if } i \textit{ is even}(\neq 2), \\ \sum_{l=0}^{n_2-1} X_2^l \varepsilon_{(i-1,j,k),\{(3,1),(3,1)\},(1,i-2),(l'_1,l'_2)} X_2^{n_2-1-l} & \textit{if } i \textit{ is odd}, \end{cases}$$
$$\varepsilon_{(i,j,k),\{(1,1),(1,1)\},(l_1,l_2),1} \mapsto$$
$$\begin{cases} -\sum_{l=0}^{n_2-1} X_2^l \varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(l_1,l_2-1),1} X_2^{n_2-1-l} & \textit{if } l_2 \textit{ is even}, \\ -\varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(l_1,l_2-1),1} X_2 + X_2 \varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(l_1,l_2-1),1} & \textit{if } l_2 \textit{ is odd}(\neq 1), \end{cases}$$
$$+ \begin{cases} X_1^{n_1-1} \varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(l_1-1,l_2),1} & \textit{if } l_1 \textit{ is even}, \\ X_1 \varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(l_1-1,l_2),1} & \textit{if } l_1 \textit{ is odd}(\neq 1), \end{cases}$$
$$\varepsilon_{(i,j,k),\{(1,1),(1,1)\},(l_1,l_2),2} \mapsto$$
$$\begin{cases} -\sum_{l=0}^{n_2-1} X_2^l \varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(l_1,l_2-1),2} X_2^{n_2-1-l} & \textit{if } l_2 \textit{ is even}, \\ -\varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(l_1,l_2-1),2} X_2 + X_2 \varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(l_1,l_2-1),2} & \textit{if } l_2 \textit{ is odd}(\neq 1), \end{cases}$$
$$+ \begin{cases} \varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(l_1-1,l_2),2} X_1^{n_1-1} & \textit{if } l_1 \textit{ is even}, \\ \varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(l_1-1,l_2),2} X_1 & \textit{if } l_1 \textit{ is odd}(\neq 1), \end{cases}$$
$$\varepsilon_{(i,j,k),\{(1,1),(1,1)\},(i-1,1),1} \mapsto$$
$$\begin{cases} -\varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(i-1,0),1} X_2 + X_2 \varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(i-1,0),1} \\ \qquad \textit{if } i \textit{ is odd and } j \textit{ is even}, \\ -\varepsilon_{(i-1,j,k),\{(1,1),(1,2)\},(i-1,0)} a_{(1,2)} X_2 + X_2 \varepsilon_{(i-1,j,k),\{(1,1),(1,2)\},(i-1,0)} a_{(1,2)} \\ \qquad \textit{if } i, j \textit{ are even}, \end{cases}$$
$$+ \begin{cases} X_1^{n_1-1} \varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(i-2,1),1} & \textit{if } i \textit{ is odd}, \\ X_1 \varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(i-2,1),1} & \textit{if } i \textit{ is even}(\neq 1), \end{cases}$$

$$\varepsilon_{(i,j,k),\{(1,1),(1,1)\},(i-1,1),2} \mapsto$$

$$\begin{cases} -\varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(i-1,0),2}X_2 + X_2\varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(i-1,0),2} \\ \qquad\qquad\qquad \textit{if } i \textit{ is odd and } j \textit{ is even}, \\ -a_{(1,1)}\varepsilon_{(i-1,j,k),\{(1,2),(1,1)\},(i-1,0)}X_2 + X_2 a_{1,1}\varepsilon_{(i-1,j,k),\{(1,2),(1,1)\},(i-1,0)} \\ \qquad\qquad\qquad \textit{if } i, j \textit{ are even}, \end{cases}$$

$$+ \begin{cases} X_1^{n_1-1}\varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(i-2,1),2} & \textit{if } i \textit{ is odd}, \\ X_1\varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(i-2,1),2} & \textit{if } i \textit{ is even}(\neq 1), \end{cases}$$

$\varepsilon_{(i,j,k),\{(1,2),(1,1)\},(i,0)} \mapsto a_{(1,2)}\varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(i-1,0),1}$ *if i is odd and j is even,*

$\varepsilon_{(i,j,k),\{(1,1),(1,2)\},(i,0)} \mapsto \varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(i-1,0),2}a_{(1,1)}$ *if i is odd and j is even,*

$\varepsilon_{(i,j,k),\{(1,1),(1,1)\},(i,0),1} \mapsto X_1^{n_1-1}a_{(1,1)}\varepsilon_{(i-1,j,k),\{(1,2),(1,1)\},(i-1,0)}$ *if i, j are even,*

$\varepsilon_{(i,j,k),\{(1,1),(1,1)\},(i,0),2} \mapsto \varepsilon_{(i-1,j,k),\{(1,1),(1,2)\},(i-1,0)}a_{(1,2)}X_1^{n_1-1}$ *if i, j are even,*

$\varepsilon_{(i,j,k),\{(3,1),(3,1)\},(i,0),1} \mapsto 0$ *if i, j are even,* ,

$\varepsilon_{(i,j,k),\{(3,1),(3,1)\},(i,0),2} \mapsto 0$ *if i, j are even,*

$\varepsilon_{(i,j,k),\{(3,2),(3,1)\},(i,0)} \mapsto 0$ *if i is even and j is odd,*

$\varepsilon_{(i,j,k),\{(3,1),(3,2)\},(i,0)} \mapsto 0$ *if i is even and j is odd.*

(7) *The following complexes depend on the complexes of* (4):

$$0 \leftarrow Q_{(i,1,k)} \xleftarrow{\partial_{(i,2,k),2}} Q_{(i,2,k)} \xleftarrow{\partial_{(i,3,k),2}} \cdots \xleftarrow{\partial_{(i,n,k),2}} Q_{(i,n,k)} \leftarrow \cdots .$$

The left A^e-homomorphism $\partial_{(i,j,k),2}$:$Q_{(i,j,k)} \to Q_{(i,j-1,k)}$ is defined as follows. We consider the case that k is odd. In the case that k is even, we have the similar result.

$\varepsilon_{(i,j,k),\{(3,1),(3,1)\},(i,0),1} \mapsto X_3^{n_3-1}a_{(3,1)}\varepsilon_{(i,j-1,k),\{(3,2),(3,1)\},(i,0)}$ *if i, j are even,* ,

$\varepsilon_{(i,j,k),\{(3,1),(3,1)\},(i,0),2} \mapsto \varepsilon_{(i,j-1,k),\{(3,1),(3,2)\},(i,0)}a_{(3,2)}X_3^{n_3-1}$ *if i, j are even,*

$\varepsilon_{(i,j,k),\{(3,2),(3,1)\},(i,0)} \mapsto a_{(3,2)}\varepsilon_{(i,j-1,k),\{(3,1),(3,1)\},(i,0),1}$ *if i is even and j is odd,*

$\varepsilon_{(i,j,k),\{(3,1),(3,2)\},(i,0)} \mapsto \varepsilon_{(i,j-1,k),\{(3,1),(3,1)\},(i,0),2}a_{(3,1)}$ *if i is even and j is odd,*

$\varepsilon_{(i,1,1),\{(3,1),(3,1)\},(i,0)} \mapsto 0$ *if $j = k = 1$.*

Then, the projective bimodule resolution of A is total complex of these complexes.

2. Main results

We have the projective bimodule resolution of A as the total complex of the complexes in Proposition 1. And, using this resolution, we determine the ring structure of the Hochschild cohomology ring of A modulo nilpotence.

Theorem 2. *We define the projective A-bimodules P_n for $n \geq 1$:*

$$P_n = \coprod_{i+j+k=n} Q_{(i,j,k)},$$

and the A^e-homomorphism d_n for $n \geq 1$:

$$\sum_{i+j=n} (\partial_{(i,j,0),1} + (-1)^{i+1}\partial_{(i,j,0),2}) + \sum_{i+j+k=n,k\geq 1} ((-1)^{j+k}\partial_{(i,j,k),1} + (-1)^{i+k+1}\partial_{(i,j,k),2} + \partial_{(i,j,k),3})$$

Then the following complex is the projective bimodule resolution of A:

$$0 \leftarrow A \xleftarrow{\pi} P_0 \xleftarrow{d_1} P_1 \xleftarrow{d_2} \cdots \xleftarrow{d_n} P_n \leftarrow \cdots .$$

Then the basis elements that are not nilpotent in Hochschild cohomology ring of A are 1_A and $e_{(2,1)} + e_{(2,2)} \in \mathrm{HH}^{2n}(A)$ for $n \geq 1$. The other elements are nilpotent elements. Therefore, we have the following result.

Theorem 3. *The Hochschild cohomology ring of A modulo nilpotence is isomorphic to the polynomial ring:*

$$\mathrm{H}^*(A)/\mathcal{N} \cong k[x]$$

where $x^n = e_{(2,1)} + e_{(2,2)} \in \mathrm{HH}^{2n}(A)$ for $n \geq 1$.

Now, we conjecture that the projective bimodule resolution of the finite dimensional algebra with quantum-like relations and monomial relations is given by the total complex of the complexes depending on the relations.

References

[1] E. L. Green, N. Snashall, Ø. Solberg, *The Hochschild cohomology ring of a selfinjective algebra of finite representation type*, Proc. Amer. Math. Soc. **131** (2003), 3387–3393.

[2] E. L. Green, N. Snashall, Ø. Solberg, *The Hochschild cohomology ring modulo nilpotence of a monomial algebra*, J. Algebra Appl. **5** (2006), 153–192.

[3] D. Obara, *Hochschild cohomology of quiver algebras defined by two cycles and a quantum-like relation*, Comm. in Algebra **40** (2012), 1724–1761.

[4] D. Obara, *Hochschild cohomology of quiver algebras defined by two cycles and a quantum-like relation* II, Comm. in Algebra **43** (2015), 3545-3587.

[5] N. Snashall, *Support varieties and the Hochschild cohomology ring modulo nilpotence*, In Proceedings of the 41st Symposium on Ring Theory and Representation Theory, Shizuoka, Japan, Sept. 5–7, 2008; Fujita, H., Ed.; Symp. Ring Theory Represent. Theory Organ. Comm., Tsukuba (2009), 68–82.

[6] N. Snashall, Ø. Solberg, *Support varieties and Hochschild cohomology rings*, Proc. London Math. Soc. **88** (2004), 705–732.

Department of Mathematics
Tokyo University of Science
1-3 Kagurazaka, Sinjuku-ku, Tokyo 162-8601 JAPAN
E-mail address: d_obara@rs.tus.ac.jp