EXAMPLES OF ORE EXTENSIONS WHICH ARE MAXIMAL ORDERS WHOSE BASED RINGS ARE NOT MAXIMAL ORDERS

H. MARUBAYASHI AND A. UEDA

Abstract. Let $R$ be a prime Goldie ring and $(\sigma, \delta)$ be a skew derivation on $R$. It is well known that if $R$ is a maximal order, then the Ore extension $R[x; \sigma, \delta]$ is a maximal order. It was a long standing open question that the converse is true or not in case $\sigma \neq 1$ and $\delta \neq 0$. We give an example of non-maximal order $R$ with a skew derivation $(\sigma, \delta)$ on $R$ ($\sigma \neq 1, \delta \neq 0$) such that $R[x; \sigma, \delta]$ is a maximal order.

1. Introduction

Let $\sigma$ be an automorphism of a ring $R$ and let $\delta$ be a left $\sigma$-derivation of $R$. Then we say $(\sigma, \delta)$ is a skew derivation on $R$. The aim of this paper is to obtain an example such that the Ore extension $R[x; \sigma, \delta]$ is a maximal order but $R$ is not a maximal order.

In case $\delta$ is trivial, the following example is known (see [1, Proposition 2.6]). Let $D$ be a hereditary Noetherian prime ring (an HNP ring for short) satisfying the following:

(a) there is a cycle $m_1, \ldots, m_n$ ($n \geq 2$) such that $m_1 \cap \cdots \cap m_n = aD = Da$ for some $a \in D$ and

(b) any maximal ideal $n$ different from $m_i$ ($1 \leq i \leq n$) is invertible.

We define a skew derivation $(\sigma, \delta)$ on $D$ by $\sigma(r) = ar a^{-1}$ and $\delta(r) = 0$ for all $r \in D$. Then $D$ is clearly not a maximal order and the Ore extension $D[x; \sigma, 0]$ is a maximal order. But in case $\sigma$ and $\delta$ are both non-trivial, we need to consider the Ore extension of a polynomial ring over $D$ and we must specify v-ideals of it.

Therefore let $R = D[t]$ be the polynomial ring over $D$ in an indeterminate $t$. Then $(\sigma, \delta)$ on $D$ is extended to a skew derivation on $R$ by $\sigma(t) = t$ and $\delta(t) = a$ (see [4, Lemma 1.2]) and it is proved that the Ore extension $R[x; \sigma, \delta]$ is maximal order but $R$ is not a maximal order (Theorem 12).

Section 2 contains preliminary results which are used in Section 3. In Section 3, we describe the structure of prime invertible ideals of $R[x; \sigma, \delta]$ (Proposition 9) and Theorem 12 is proved by showing that any v-ideal is v-invertible.

We refer the readers to [12] and [13] for terminology not defined in the paper.

2. Preliminary results

Let $S$ be a Noetherian prime ring with quotient ring $Q$ and $A$ be a fractional $S$-ideal. We use the following notation:

$$ (S : A)_l = \{ q \in Q \mid qA \subseteq S \}, \quad (S : A)_r = \{ q \in Q \mid Aq \subseteq S \} $$

$$ A_v = (S : (S : A)_l)_r \supseteq A \quad \text{and} \quad vA = (S : (S : A)_r)_l \supseteq A. $$

The detailed version of this paper will be submitted for publication elsewhere.
A is called a \( v \)-ideal if \( vA = A = A_v \). A \( v \)-ideal \( A \) is said to be \( v \)-invertible (invertible) if \( (S : A)_rA = S = (A(S : A)_r)_r \) and \( (S : A)_r^{-1} = (S : A)_r \), respectively.

Note that if \( A \) is \( v \)-invertible, then it is easy to see that \( O_v(A) = S = O_l(A) \) and \( (S : A)_r = A^{-1} = (S : A)_r \), where \( O_l(A) = \{ q \in Q \mid qA \subseteq A \} \), a left order of \( A \), \( O_v(A) = \{ q \in Q \mid Aq \subseteq A \} \), a right order of \( A \) and \( A^{-1} = \{ q \in Q \mid AqA \subseteq A \} \).

Concerning invertible ideals and \( v \)-invertible ideals of \( S \), the next lemma holds.

**Lemma 1.** A \( v \)-ideal is invertible if and only if it is \( v \)-invertible and projective (left and right projective).

In the remainder of this section, let \( D \) be a hereditary Noetherian prime ring (an HNP ring for short) with quotient ring \( K = Q(D) \) and \( R = D[t] \). Let \( \sigma \) be an inner automorphism induced by a regular element \( a \) of \( D \), that is, \( \sigma(r) = ara^{-1} \) for all \( r \in D \) and \( \delta \) be a trivial left \( \sigma \)-derivation on \( D \), that is, \( \delta(r) = 0 \) for all \( r \in D \).

Put \( R = D[t] \), the polynomial ring over \( D \) in an indeterminate \( t \). \( \sigma \) and \( \delta \) are extended to an automorphism \( \sigma \) of \( R \) and a left \( \sigma \)-derivation \( \delta \) on \( R \) as follows ([4, Lemma 1.2]);

\[
\sigma(t) = t \quad \text{and} \quad \delta(t) = a.
\]

It is well-known that a skew derivation \( (\sigma, \delta) \) is naturally extended to a skew derivation on \( K \) ([12, p. 132]). Also we note that \( \sigma\delta = \delta\sigma \) holds.

We put

\[
V_r(R) = \{ a : \text{ ideals } \mid a = a_v \} \supseteq V_{(m,r)}(R) = \{ a \in V_r(R) \mid a \text{ is maximal in } V_r(R) \},
\]

\[
V_l(R) = \{ a : \text{ ideals } \mid a = v_a \} \supseteq V_{(m,l)}(R) = \{ a \in V_l(R) \mid a \text{ is maximal in } V_l(R) \}
\]

and

\[
\text{Spec}_0(R) = \{ b : \text{ prime ideals } \mid b \cap D = (0) \text{ and } b \text{ is a } v \text{-ideal} \}.
\]

Note that for each fractional \( R \)-ideal \( a, a = a_v \) if and only if \( a \) is right projective by [2, Proposition 5.2] and that there is a one-to-one correspondence between \( \text{Spec}_0(R) \) and \( \text{Spec}(K[t]) \) (see [12, Proposition 2.3.17]).

Using these facts, we can prove the following lemma.

**Lemma 2.** \( V_{(m,r)}(R) = V_{(m,t)}(R) \) and is equal to

\[
V_{m}(R) = \{ m[t], b \mid m \text{ runs over all maximal ideals of } D \text{ and } b \in \text{Spec}_0(R) \}.
\]

From Lemmas 1 and 2, we have the following.

**Lemma 3.** If \( b \in \text{Spec}_0(R) \), then \( b \) is invertible.

Now we can determine the maximal invertible ideals of \( R \) by Lemmas 2 and 3.

**Proposition 4.** \( \{ p[t] = m_1[t] \cap \cdots \cap m_k[t], b \mid m_1, \ldots, m_k \text{ is a cycle of } D, k \geq 1, b \in \text{Spec}_0(R) \} \) is the full set of maximal invertible ideals of \( R \) (ideals maximal amongst the invertible ideals).

The following proposition follows from the proof of [3, Proposition 2.1 and Theorem 2.9].

**Proposition 5.** The invertible ideals of \( R \) generate an Abelian group whose generators are maximal invertible ideals.
In case $D$ has enough invertible ideals, it is shown in [9] that $R = D[t]$ is a $v$-HC order with enough $v$-invertible ideals, which is a Krull type generalization of HNP rings. Recall the notion of $v$-HC orders: A Noetherian prime ring $S$ is called a $v$-HC order if $v(A(S : A)) = O_v(A)$ for any ideal $A$ of $S$ with $A = vA$ and $((R : S), B)_v = O_v(B)$ for any ideal $B$ of $S$ with $B = B_v$. A $v$-HC order $S$ is said to be having enough $v$-invertible ideals if any $v$-ideal of $S$ contains a $v$-ideal which is $v$-invertible. A $v$-ideal $C$ is called eventually $v$-idempotent if $(C^n)_v$ is $v$-idempotent for some $n \geq 1$, that is, $((C^n)_v^2)_v = (C^n)_v$.

Ideal theory in HNP rings are generalized to one in $v$-HC orders with enough $v$-invertible ideals. The following two lemmas are very useful to investigate the structure of $v$-ideals of $v$-HC orders (for their proofs, see [8, Lemma 1.1] and [10, Lemma 1 and Proposition 3]).

**Lemma 6.** Let $S$ be a prime Goldie ring and $A, B$ be fractional $S$-ideals.

1. $(AB)_v = (AB)_v$.
2. $(A, B)_v = (A)_v$ if $B$ is $v$-invertible.
3. $(AB)_v = A_vB$ if $B$ is invertible.

**Lemma 7.** Let $S$ be a $v$-HC order with enough $v$-invertible ideals and $A$ be a fractional $S$-ideal.

1. $vA = A_v$.
2. $A_v = (BC)_v$ for some $v$-invertible ideal $B$ and eventually $v$-idempotent ideal $C$.
3. Let $C$ be an eventually $v$-idempotent ideal and let $M_1, \ldots, M_k$ be the full set of maximal $v$-ideals containing $C$. Then $(C^k)_v = ((M_1 \cap \cdots \cap M_k)_v^k)$ and is $v$-idempotent.

**Remark.** A $v$-ideal of $S$ is eventually $v$-idempotent if and only if it is not contained in any $v$-invertible ideals (see the proofs of [3, Propositions 4.3 and 4.5]).

3. **Examples**

Throughout this section, $D$ is an HNP ring with quotient ring $K$ satisfying the following:

(a) there is a cycle $m_1, \ldots, m_n$ such that $m_1 \cap \cdots \cap m_n = aD = Da$ for some $a \in D$.
(b) any maximal ideal $n$ different from $m_i$ ($1 \leq i \leq n$) is invertible.

Examples of an HNP ring $D$ satisfying the conditions (a) and (b) are found in [6] and [1]. The simplest example is $D = \left( \begin{array}{cc} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{array} \right)$, where $\mathbb{Z}$ is the ring of integers and $p$ is a prime number.

Unless otherwise stated, $R = D[t]$, $\sigma$ is an automorphism of $R$ and $\delta$ is a left $\sigma$-derivation as in Section 1, that is, $\sigma(r) = ar^{-1}$, $\delta(r) = 0$ for all $r \in D$, $\sigma(t) = t$ and $\delta(t) = a$.

Note that $\sigma(m_i) = m_{i+1}$ ($1 \leq i \leq n - 1$), $\sigma(m_n) = m_1$ and $\sigma(n) = n$ for all maximal ideals $n$ with $n \neq m_i$ ($1 \leq i \leq n$) by [5, Theorem 14] and [9, Corollary 2.3]. Furthermore, by Lemma 2 and Proposition 4,

$$V_m(R) = \{m_i[t], n[t], b \mid n \neq m_i \text{ and } b \in \text{Spec}_0(R)\}$$

and

$$I_m(R) = \{p[t], n[t], b \mid p = m_1 \cap \cdots \cap m_n, n \neq m_i \text{ and } b \in \text{Spec}_0(R)\}$$

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is the set of all maximal invertible ideals of $R$.

Note that a maximal ideal of $K[t]$ is either $tK[t]$ or $\omega K[t]$ for some $\omega = k_1 t^l + \cdots + k_0 \in Z(K[t])$ with $k_l \neq 0$, $k_0 \neq 0$, $l \geq 1$, where $Z(K[t])$ is the center of $K[t]$ (see [12, Theorem 2.3.10]) and so any $b \in \Spec_0(R)$ is either $t R$ or $b = \omega K[t] \cap R$, where $\omega \in Z(K[t])$ and $\omega K[t]$ is a maximal ideal ([12, Proposition 2.3.17]).

A fractional $R$-ideal $a$ is called $\sigma$-invariant if $\sigma(a) = a$ and is called $\delta$-stable if $\delta(a) \subseteq a$. A $\sigma$-invariant and $\delta$-stable fractional $R$-ideal is said to be $(\sigma, \delta)$-stable.

The following lemma is crucial to study ideals of $R$ and is proved by using the results obtained in section 2.

**Lemma 8.**

1. Any projective ideal of $R$ is a product of an invertible ideal and an eventually $v$-idempotent ideal.
2. Any eventually $v$-idempotent ideal is not $\sigma$-invariant.
3. $n[t]$ and $p[t]$ are $(\sigma, \delta)$-stable.
4. Let $\omega = t$ or $\omega \in Z(K[t])$ and let $b = \omega K[t] \cap R$, which is a maximal invertible ideal of $R$. Then
   
   i. $b^n$ is $\sigma$-invariant for any $n \geq 1$.
   
   ii. $b^n$ is $\delta$-stable if and only if $\omega^n K[t]$ is $\delta$-stable if and only if $\delta(\omega^n) = 0$.
   
   iii. (a) If $\char K = 0$, then $b^n$ is not $\delta$-stable for any $n$.
   
   (b) If $\char K = p \neq 0$ and $\delta(\omega) \neq 0$, then $b^p$ is $(\sigma, \delta)$-stable and $b^i$ is not $(\sigma, \delta)$-stable $(1 \leq i < p)$.
   
   (c) If $\char K = p \neq 0$ and $\delta(\omega) = 0$, then $b^n$ is $(\sigma, \delta)$-stable for all $n \geq 1$.

In the remainder of this section, let $S = R[x; \sigma, \delta]$, an Ore extension in an indeterminate $x$ and $T = Q[x; \sigma, \delta]$, where $Q = Q(R)$, the quotient ring of $R$. We will prove that $S$ is a maximal order. To prove maximality of $S$, it is enough to show that each $v$-ideal of $S$ is $v$-invertible. For this purpose, we will describe all $v$-ideals of $S$.

Note that for an ideal $a$ of $R$, $a[x; \sigma, \delta]$ is an ideal of $S$ if and only if $a$ is $(\sigma, \delta)$-stable.

From Lemma 8, we have the following Proposition 9 and we can prove invertibility of a $v$-ideal $A$ of $S$ such as $A \cap R \neq (0)$ by using Proposition 9.

**Proposition 9.** Under the same notations as in Lemma 8, let $A$ be an ideal of $S$ such that $A = A_v$ and is maximal in $\{ B : \text{ideal} \mid B = B_v \}$. If $A \cap R = a \neq (0)$, then $A$ is equal to one of $P = p[t][x; \sigma, \delta]$, $N = n[t][x; \sigma, \delta]$, $B = b[x; \sigma, \delta]$ (in case $b$ is $(\sigma, \delta)$-stable) or $C = b^p[x; \sigma, \delta]$ (in case $b$ is $\sigma$-invariant but not $\delta$-stable) and each of these is a prime $v$-invertible ideal of $S$.

**Lemma 10.** Let $A$ be an ideal of $S$ such that $A = A_v$ and $a = A \cap R \neq (0)$. Then $a$ is a $(\sigma, \delta)$-stable invertible ideal and $A = a[x; \sigma, \delta]$.

**Outline of Proof.** Assume that $A \supset a[x; \sigma, \delta]$ and that it is maximal for this property. Then, by Proposition 9, there is a $P_0 = p_0[t][x; \sigma, \delta] \supset A$, where $p_0 = p[t]$ or $n[t]$ or $b$ or $c$ and $S \supset AP_0^{-1} \supset A$. Then $AP_0^{-1} = a'[x; \sigma, \delta]$ for some $(\sigma, \delta)$-stable $v$-ideal $a'$, and $A = ((AP_0^{-1})P_0)_v = (a'p_0)_v[x; \sigma, \delta]$, which is a contradiction. $\square$

By Lemma 10, we can prove also $v$-invertibility of a $v$-ideal $A$ such as $A \cap R = (0)$.

**Lemma 11.** Let $A$ be an ideal of $S$ such that $A = A_v$ and $A \cap R = (0)$. Then $A$ is $v$-invertible.
Outline of Proof. \( T = (S : A)_iAT \) holds and so \((S : A)_iA \cap R \neq (0)\). Then \( v((S : A)_iA) \) is invertible by the left version of Lemma 10. Suppose \( v((S : A)_iA) \subseteq S \). Then there is a maximal invertible ideal \( P_0 \) which is prime and \( P_0 \geq v((S : A)_iA) \). Then the localization \( S_{P_0} \) is a local Dedekind prime ring

\[
S_{P_0} = (S_{P_0} : AS_{P_0})_iAS_{P_0} \subseteq S_{P_0}(S : A)_iAS_{P_0} \subseteq S_{P_0}P_0S_{P_0} = J(S_{P_0}),
\]

the Jacobson radical of \( S_{P_0} \), which is a contradiction.

Now we obtain the main theorem of this paper by Lemmas 10 and 11.

**Theorem 12.** \( S = R[x; \sigma, \delta] \) is a maximal order and \( R \) is not a maximal order.

**Proof.** Let \( A \) be any non-zero ideal of \( S \). Since \( S \subseteq O_l(A) \subseteq O_l(A_v) \), in order to prove \( O_l(A) = S \), we may assume that \( A = A_v \). By Lemmas 10 and 11, \( A \) is \( (v) \)-invertible. Hence \( O_l(A) = S \) and similarly \( O_r(A) = S \), that is, \( S \) is a maximal order. Of course \( R \) is not a maximal order.

As an application of Theorem 12, we give the example related to unique factorization rings. A Noetherian prime ring \( R \) is called a *unique factorization ring* (a UFR for short) if each prime ideal \( P \) with \( P = P_v \) (or \( P = vP \)) is principal, that is, \( P = bR = Rb \) for some \( b \in R \). We note that \( R \) is a UFR if and only if \( R \) is a maximal order and each \( v \)-ideal is principal, and if \( R \) is a maximal order, then every prime \( v \)-ideal is a maximal \( v \)-ideal.

Then we have the following.

**Proposition 13.** Suppose \( \text{char} \, D = 0 \) and any maximal ideal \( \mathfrak{m} \) different from \( \mathfrak{m}_i \) \((1 \leq i \leq n)\) is principal. Then \( S = R[x; \sigma, \delta] \) is a UFR but \( R \) is not a UFR.

At the end, we state an open problem concerning Ore extensions.

**Problem.** Let \( R \) be a prime Goldie ring and consider the Ore extension \( R[x; \sigma, \delta] \) of \( R \), where \((\sigma, \delta)\) is a skew derivation on \( R \). Then what is necessary and sufficient condition for \( R[x; \sigma, \delta] \) to be a maximal order or unique factorization ring?

**References**


Faculty of Sciences and Engineering
Tokushima Bunri University
Sanuki, Kagawa, 769-2193, JAPAN
E-mail address: marubaya@kagawa.bunri-u.ac.jp

Department of Mathematics
Shimane University
Matsue, Shimane, 690-8504, JAPAN
E-mail address: ueda@riko.shimane-u.ac.jp