EXAPMLES OF ORE EXTENSIONS WHICH ARE MAXIMAL ORDERS WHOSE BASED RINGS ARE NOT MAXIMAL ORDERS

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ABSTRACT. Let R be a prime Goldie ring and (σ, δ) be a skew derivation on R. It is well known that if R is a maximal order, then the Ore extension $R[x; \sigma, \delta]$ is a maximal order. It was a long standing open question that the convserse is true or not in case $\sigma \neq 1$ and $\delta \neq 0$. We give an example of non-maximal order R with a skew derivation (σ, δ) on R ($\sigma \neq 1, \delta \neq 0$) such that $R[x; \sigma, \delta]$ is a maximal order.

1. Introduction

Let σ be an automorphism of a ring R and let δ be a left σ-derivation of R. Then we say (σ, δ) is a skew derivation on R. The aim of this paper is to obtain an example such that the Ore extension $R[x; \sigma, \delta]$ is a maximal order but R is not a maximal order.

In case δ is trivial, the following example is known (see [1, Proposition 2.6]). Let D be a hereditary Noetherian prime ring (an HNP ring for short) satisfying the following:

(a) there is a cycle $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ ($n \geq 2$) such that $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n = aD = Da$ for some $a \in D$ and

(b) any maximal ideal $\mathfrak{n}$ different from $\mathfrak{m}_i$ ($1 \leq i \leq n$) is invertible.

We define a skew derivation (σ, δ) on D by $\sigma(r) = ara^{-1}$ and $\delta(r) = 0$ for all $r \in D$. Then D is clearly not a maximal order and the Ore extension $D[x; \sigma, 0]$ is a maximal order. But in case σ and δ are both non-tirvial, we need to consider the Ore extension of a polynomial ring over D and we must specify v-ideals of it.

Therefore let $R = D[t]$ be the polynomial ring over D in an indeterminate t. Then (σ, δ) on D is extended to a skew derivation on R by $\sigma(t) = t$ and $\delta(t) = a$ (see [4, Lemma 1.2]) and it is proved that the Ore extension $R[x; \sigma, \delta]$ is maximal order but R is not a maximal order (Theorem 12).

Section 2 contains preliminary results which are used in Section 3. In Section 3, we describe the structure of prime invertible ideals of $R[x; \sigma, \delta]$ (Proposition 9) and Theorem 12 is proved by showing that any v-ideal is v-invertible.

We refer the readers to [12] and [13] for terminology not defined in the paper.

2. Preliminary results

Let S be a Noetherian prime ring with quotient ring Q and A be a fractional S-ideal. We use the following notation:

$$(S : A)_l = \{q \in Q \mid qA \subseteq S\}, \quad (S : A)_r = \{q \in Q \mid Aq \subseteq S\} \text{ and}$$
$$A_v = (S : (S : A)_l)_r \supseteq A \quad \text{and} \quad {}_vA = (S : (S : A)_r)_l \supseteq A.$$

The detailed version of this paper will be submitted for publication elsewhere.

A is called a *v-ideal* if ${}_vA = A = A_v$. A v-ideal A is said to be *v-invertible* (*invertible*) if ${}_v((S:A)_lA) = S = (A(S:A)_r)_v$ $((S:A)_lA = S = A(S:A)_r)$, respectively.

Note that if A is v-invertible, then it is easy to see that $O_r(A) = S = O_l(A)$ and $(S:A)_l = A^{-1} = (S:A)_r$, where $O_l(A) = \{q \in Q \mid qA \subseteq A\}$, a *left order* of A, $O_r(A) = \{q \in Q \mid Aq \subseteq A\}$, a *right order* of A and $A^{-1} = \{q \in Q \mid AqA \subseteq A\}$.

Concerning invertible ideals and v-invertible ideals of S, the next lemma holds.

Lemma 1. *A v-ideal is invertible if and only if it is v-invertible and projective (left and right projective).*

In the remainder of this section, *let D be a hereditary Noetherian prime ring (an HNP ring for short) with quotient ring $K = Q(D)$ and $R = D[t]$*. Let σ be an inner automorphism induced by a regular element a of D, that is, $\sigma(r) = ara^{-1}$ for all $r \in D$ and δ be a trivial left σ-derivation on D, that is, $\delta(r) = 0$ for all $r \in D$.

Put $R = D[t]$, the polynomial ring over D in an indeterminate t. σ and δ are extended to an automorphism σ of R and a left σ-derivation δ on R as follows ([4, Lemma 1.2]);

$$\sigma(t) = t \quad \text{and} \quad \delta(t) = a.$$

It is well-known that a skew derivation (σ, δ) is naturally extended to a skew derivation on K ([12, p. 132]). Also we note that $\sigma\delta = \delta\sigma$ holds.

We put

$$\begin{aligned}
V_r(R) &= \{\mathfrak{a} : \text{ ideals } \mid \mathfrak{a} = \mathfrak{a}_v\} \supseteq V_{(m,r)}(R) = \{\mathfrak{a} \in V_r(R) \mid \mathfrak{a} \text{ is maximal in } V_r(R)\},\\
V_l(R) &= \{\mathfrak{a} : \text{ ideals } \mid \mathfrak{a} = {}_v\mathfrak{a}\} \supseteq V_{(m,l)}(R) = \{\mathfrak{a} \in V_l(R) \mid \mathfrak{a} \text{ is maximal in } V_l(R)\} \text{ and}\\
\mathrm{Spec}_0(R) &= \{\mathfrak{b} : \text{ prime ideals } \mid \mathfrak{b} \cap D = (0) \text{ and } \mathfrak{b} \text{ is a v-ideal}\}.
\end{aligned}$$

Note that for each fractional R-ideal $\mathfrak{a}$, $\mathfrak{a} = \mathfrak{a}_v$ if and only if $\mathfrak{a}$ is right projective by [2, Proposition 5.2] and that there is a one-to-one correspondence between $\mathrm{Spec}_0(R)$ and $\mathrm{Spec}(K[t])$ (see [12, Proposition 2.3.17]).

Using these facts, we can prove the following lemma.

Lemma 2. $V_{(m,r)}(R) = V_{(m,l)}(R)$ *and is equal to*

$$V_m(R) = \{\mathfrak{m}[t], \mathfrak{b} \mid \mathfrak{m} \textit{ runs over all maximal ideals of } D \textit{ and } \mathfrak{b} \in \mathit{Spec}_0(R)\}.$$

From Lemmas 1 and 2, we have the following.

Lemma 3. *If $\mathfrak{b} \in Spec_0(R)$, then $\mathfrak{b}$ is invertible.*

Now we can determine the maximal invertible ideals of R by Lemmas 2 and 3.

Proposition 4. $\{\mathfrak{p}[t] = \mathfrak{m}_1[t] \cap \cdots \cap \mathfrak{m}_k[t], \mathfrak{b} \mid \mathfrak{m}_1, \ldots, \mathfrak{m}_k$ *is a cycle of* $D, k \geq 1, \mathfrak{b} \in$ *$Spec_0(R)\}$ is the full set of maximal invertible ideals of R (ideals maximal amongst the invertible ideals).*

The following proposition follows from the proof of [3, Proposition 2.1 and Theorem 2.9].

Proposition 5. *The invertible ideals of R generate an Abelian group whose generators are maximal invertible ideals.*

In case D has enough invertible ideals, it is shown in [9] that $R = D[t]$ is a v-HC order with enough v-invertible ideals, which is a Krull type generalization of HNP rings. Recall the notion of v-HC orders: A Noetherian prime ring S is called a *v-HC order* if ${}_v(A(S : A)_l) = O_l(A)$ for any ideal A of S with $A = {}_vA$ and $((R : S)_r B)_v = O_r(B)$ for any ideal B of S with $B = B_v$. A v-HC order S is said to be *having enough v-invertible ideals* if any v-ideal of S contains a v-ideal which is v-invertible. A v-ideal C is called *eventually v-idempotent* if $(C^n)_v$ is v-idempotent for some $n \geq 1$, that is, $(({(C^n)_v}^2)_v = (C^n)_v$.

Ideal theory in HNP rings are generalized to one in v-HC orders with enough v-invertible ideals. The following two lemmas are very useful to investigate the structure of v-ideals of v-HC orders (for their proofs, see [8, Lemma 1.1] and [10, Lemma 1 and Proposition 3]).

Lemma 6. *Let S be a prime Goldie ring and A, B be fractional S-ideals.*

(1) $(AB)_v = (AB_v)_v$.
(2) $(A_vB)_v = (AB)_v$ *if B is v-invertible.*
(3) $(AB)_v = A_vB$ *if B is invertible.*

Lemma 7. *Let S be a v-HC order with enough v-invertible ideals and A be a fractional S-ideal.*

(1) ${}_vA = A_v$.
(2) $A_v = (BC)_v$ *for some v-invertible ideal B and eventually v-idempotent ideal C.*
(3) *Let C be an eventually v-idempotent ideal and let $M_1, \dots, M_k$ be the full set of maximal v-ideals containing C. Then $(C^k)_v = ((M_1 \cap \cdots \cap M_k)^k)_v$ and is v-idempotent.*

Remark. A v-ideal of S is eventually v-idempotent if and only if it is not contained in any v-invertible ideals (see the proofs of [3, Propositions 4.3 and 4.5]).

3. Examples

Throughout this section, *D is an HNP ring with quotient ring K satisfying the following*:

(a) there is a cycle $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ such that $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n = aD = Da$ for some $a \in D$.
(b) any maximal ideal $\mathfrak{n}$ different from $\mathfrak{m}_i$ $(1 \leq i \leq n)$ is invertible.

Examples of an HNP ring D satisfying the conditions (a) and (b) are found in [6] and [1]. The simplest example is $D = \begin{pmatrix} \mathbb{Z} & p\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$, where $\mathbb{Z}$ is the ring of integers and p is a prime number.

Unless otherwise stated, $R = D[t]$, σ is an automorphism of R and δ is a left σ-derivation as in Section 1, that is, $\sigma(r) = ara^{-1}$, $\delta(r) = 0$ for all $r \in D$, $\sigma(t) = t$ and $\delta(t) = a$.

Note that $\sigma(\mathfrak{m}_i) = \mathfrak{m}_{i+1}$ $(1 \leq i \leq n-1)$, $\sigma(\mathfrak{m}_n) = \mathfrak{m}_1$ and $\sigma(\mathfrak{n}) = \mathfrak{n}$ for all maximal ideals $\mathfrak{n}$ with $\mathfrak{n} \neq \mathfrak{m}_i$ $(1 \leq i \leq n)$ by [5, Theorem 14] and [9, Corollary 2.3]. Furthermore, by Lemma 2 and Proposition 4,

$$V_m(R) = \{\mathfrak{m}_i[t],\ \mathfrak{n}[t],\ \mathfrak{b} \mid \mathfrak{n} \neq \mathfrak{m}_i \text{ and } \mathfrak{b} \in \operatorname{Spec}_0(R)\}$$

and

$$I_m(R) = \{\mathfrak{p}[t],\ \mathfrak{n}[t],\ \mathfrak{b} \mid \mathfrak{p} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n, \mathfrak{n} \neq \mathfrak{m}_i \text{ and } \mathfrak{b} \in \operatorname{Spec}_0(R)\}$$

is the set of all maximal invertible ideals of R.

Note that a maximal ideal of $K[t]$ is either $tK[t]$ or $\omega K[t]$ for some $\omega = k_l t^l + \cdots + k_0 \in Z(K[t])$ with $k_l \neq 0, k_0 \neq 0, l \geq 1$, where $Z(K[t])$ is the center of $K[t]$ (see [12, Theorem 2.3.10]) and so any $\mathfrak{b} \in \mathrm{Spec}_0(R)$ is either $\mathfrak{b} = tR$ or $\mathfrak{b} = \omega K[t] \cap R$, where $\omega \in Z(K[t])$ and $\omega K[t]$ is a maximal ideal ([12, Proposition 2.3.17]).

A fractional R-ideal $\mathfrak{a}$ is called *σ-invariant* if $\sigma(\mathfrak{a}) = \mathfrak{a}$ and is called *δ-stable* if $\delta(\mathfrak{a}) \subseteq \mathfrak{a}$. A σ-invariant and δ-stable fractional R-ideal is said to be *(σ, δ)-stable.*

The following lemma is crutial to study ideals of R and is proved by using the results obtained in section 2.

Lemma 8. (1) *Any projective ideal of R is a product of an invertible ideal and an eventually v-idempotent ideal.*
(2) *Any eventually v-idempotent ideal is not σ-invariant.*
(3) *$\mathfrak{n}[t]$ and $\mathfrak{p}[t]$ are (σ, δ)-stable.*
(4) *Let $\omega = t$ or $\omega \in Z(K[t])$ and let $\mathfrak{b} = \omega K[t] \cap R$, which is a maximal invertible ideal of R. Then*
  (i) *$\mathfrak{b}^n$ is σ-invariant for any $n \geq 1$.*
  (ii) *$\mathfrak{b}^n$ is δ-stable if and only if $\omega^n K[t]$ is δ-stable if and only if $\delta(\omega^n) = 0$.*
  (iii) (a) *If char $K = 0$, then $\mathfrak{b}^n$ is not δ-stable for any n.*
    (b) *If char $K = p \neq 0$ and $\delta(\omega) \neq 0$, then $\mathfrak{b}^p$ is (σ, δ)-stable and $\mathfrak{b}^i$ is not (σ, δ)-stable $(1 \leq i < p)$.*
    (c) *If char $K = p \neq 0$ and $\delta(\omega) = 0$, then $\mathfrak{b}^n$ is (σ, δ)-stable for all $n \geq 1$.*

In the remainder of this section, *let $S = R[x; \sigma, \delta]$, an Ore extension in an indeterminate x and $T = Q[x; \sigma, \delta]$, where $Q = Q(R)$, the quotient ring of R.* We will prove that S is a maximal order. To prove maximality of S, it is enough to show that each v-ideal of S is v-invertible. For this purpose, we will describe all v-ideals of S.

Note that for an ideal $\mathfrak{a}$ of R, $\mathfrak{a}[x; \sigma, \delta]$ is an ideal of S if and only if $\mathfrak{a}$ is (σ, δ)-stable.

From Lemma 8, we have the following Proposition 9 and we can prove invertibility of a v-ideal A of S such as $A \cap R \neq (0)$ by using Proposition 9.

Proposition 9. *Under the same notations as in Lemma 8, let A be an ideal of S such that $A = A_v$ and is maximal in $\{B : \text{ ideal } \mid B = B_v\}$. If $A \cap R = \mathfrak{a} \neq (0)$, then A is equal to one of $P = \mathfrak{p}[t][x; \sigma, \delta]$, $N = \mathfrak{n}[t][x; \sigma, \delta]$, $B = \mathfrak{b}[x; \sigma, \delta]$ (in case $\mathfrak{b}$ is (σ, δ)-stable) or $C = \mathfrak{b}^p[x; \sigma, \delta]$ (in case $\mathfrak{b}$ is σ-invariant but not δ-stable) and each of these is a prime invertible ideal of S.*

Lemma 10. *Let A be an ideal of S such that $A = A_v$ and $\mathfrak{a} = A \cap R \neq (0)$. Then $\mathfrak{a}$ is a (σ, δ)-stable invertible ideal and $A = \mathfrak{a}[x; \sigma, \delta]$.*

Outline of Proof. Assume that $A \supset \mathfrak{a}[x; \sigma, \delta]$ and that it is maximal for this property. Then, by Proposition 9, there is a $P_0 = \mathfrak{p}_0[x; \sigma, \delta] \supset A$, where $\mathfrak{p}_0 = \mathfrak{p}_[t]$ or $\mathfrak{n}[t]$ or $\mathfrak{b}$ or $\mathfrak{c}$ and $S \supseteq AP_0^{-1} \supset A$. Then $AP_0^{-1} = \mathfrak{a}'[x; \sigma, \delta]$ for some (σ, δ)-stable v-ideal $\mathfrak{a}'$, and $A = ((AP_0^{-1})P_0)_v = (\mathfrak{a}'\mathfrak{p}_0)_v[x; \sigma, \delta]$, which is a contradiction. □

By Lemma 10, we can prove also v-invertibility of a v-ideal A such as $A \cap R = (0)$.

Lemma 11. *Let A be an ideal of S such that $A = A_v$ and $A \cap R = (0)$. Then A is v-invertible.*

Outline of Proof. $T = (S : A)_l AT$ holds and so $(S : A)_l A \cap R \neq (0)$. Then $_v((S : A)_l A)$ is invertible by the left version of Lemma 10. Suppose $_v((S : A)_l A) \subset S$. Then there is a maximal invertible ideal P_0 which is prime and $P_0 \supseteq {}_v((S : A)_l A)$. Then the localizaion S_{P_0} is a local Dedekind prime ring and

$$S_{P_0} = (S_{P_0} : AS_{P_0})_l AS_{P_0} \subseteq S_{P_0}(S : A)_l AS_{P_0} \subseteq S_{P_0} P_0 S_{P_0} = J(S_{P_0}),$$

the Jacobson radical of S_{P_0}, which is a contradiction. □

Now we obtain the main theorem of this paper by Lemmas 10 and 11.

Theorem 12. *$S = R[x; \sigma, \delta]$ is a maximal order and R is not a maximal order.*

Proof. Let A be any non-zero ideal of S. Since $S \subseteq O_l(A) \subseteq O_l(A_v)$, in order to prove $O_l(A) = S$, we may assume that $A = A_v$. By Lemmas 10 and 11, A is (v)-invertible . Hence $O_l(A) = S$ and similarly $O_r(A) = S$, that is, S is a maximal order. Of course R is not a maximal order. □

As an application of Theorem 12, we give the example related to unique factorization rings. A Noetherian prime ring R is called a *unique factorization ring* (a UFR for short) if each prime ideal P with $P = P_v$ (or $P = {}_vP$) is principal, that is, $P = bR = Rb$ for some $b \in R$. We note that R is a UFR if and only if R is a maximal order and each v-ideal is principal, and if R is a maximal order, then every prime v-ideal is a maximal v-ideal.

Then we have the following.

Proposition 13. *Suppose char $D = 0$ and any maximal ideal $\mathfrak{n}$ different from $\mathfrak{m}_i$ ($1 \leq i \leq n$) is principal. Then $S = R[x; \sigma, \delta]$ is a UFR but R is not a UFR.*

At the end, we state an open problem concerning Ore extensions.

Problem. Let R be a prime Goldie ring and consider the Ore extension $R[x; \sigma, \delta]$ of R, where (σ, δ) is a skew derivation on R. Then what is necessary and sufficient condition for $R[x; \sigma, \delta]$ to be a maximal order or unique factorization ring?

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