QUIVERS, OPERATORS ON HILBERT SPACES AND OPERATOR ALGEBRAS

YASUO WATATANI (綿谷　安男）

Abstract. One of the aims of the theory of representations of finite dimensional algebras is to describe how linear transformations can act simultaneously on a finite dimensional vector space. We consider bounded linear operators on a infinite-dimensional Hilbert space instead of linear transformations on a finite dimensional vector space. We describe similarities and differences between ring theory and theory of operator algebras.

1. Introduction

Operator algebraists import many ideas from ring theory without paying anything. Ring theorists import few ideas from theory of operator algebras because it is based on functional analysis. But we expect more fruitful interactions between these two theory. For example, quivers are related with operator algebras in the following (at least) three different stages:
(1)Cuntz-Krieger algebras [1]
(2)Principal graphs for subfactors [5], [4],[8]
(3)Hilbert representations of quivers [2], [3]

First We describe similarities and differences between ring theory and theory of operator algebras.

(1)Cuntz-Krieger algebras [1]

Strongly connected quivers generate an important class of purely infinite simple C^*-algebras, called Cuntz-Krieger algebras, and they are classified by their K-groups. The vertices are represented by orthogonal subspaces and the arrows are represented by partial isometries with the orthognal ranges.

(2)Principal graphs for subfactors [5], [4],[8]

The category of bimodules for a subfactor forms a principal graph (a certain bitartite graph) and a good invariant in subfactor theoory. In particular, irreducible hyperfinite subfactors with Jones index less than four have Dynkin diagrams A,D and E. The vertices are constructed by irreducible bimodules and arrows are constructed by bimodule homomorphisms.

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Ring Theory, Representation Theory	*Theory of Operator algebras*
Algebra	Functional analysis
Finite dimensional alg. are improtant	Finite dimensional alg. are trivial
Infinite dimensional alg. are interesting	Infinite dimensional alg. are essential
vector space	Hilbert space (need inner product)
ring	*-algebra
non-commutative	non-commutative
over a field K	over $\mathbb{C}$
no measure theory	need measure theory
category is an essential tool	category is a useful language
combinatrics	topotopological approximation
?	positivity
charateristic p	?
polynomial ring	algebra of continuous functions
path algebra	Cuntz-Krieger algebra

(3)Hilbert representations of quivers [2], [3]

A Hilbert representation of a quiver is to associate Hilbert spaces for the vertices and bounded operators for arrows. Jordan blocks correspond to strongly irreducible operators. The invaraint subspace problem is one of the famous unsolved problems in functional anaysis and rephrased by the existence of a simple Hilbert representation of a quiver.

We study operator algebras instead of finite dimensional algebras. We have two important classes of operator algebras, that is, C^*-algebras and von Neumann algebras. A C^*-algebra is a *-subalgebra of the algebra $B(H)$ of bounded operators on a Hilbert space H closed under operator-norm-topology. A von Neumann algebra is a *-subalgebra of $B(H)$ closed under weak-operator-topology. C^*-algebras are regarded as quantized (locally) compact Housdorff spaces. Von Neumann algebras are regarded as quantized measure spaces.

We can associated C^*-algebras for topological dynamical systems and von Neumann algebras for measurable dynamical systems. In the half of this note, we will show our study on C^*-algebras associated complex dynamical systems ([6]) and self-similar dynamical systems ([7]) and on Hilbert representations. These results are based on joint works with Tsuyoshi Kajiwara and Masatoshi Enomoto.

In order to "feel" the difference between purely algebraic setting and functional analytic setting, let us consider the following typical examples: Let L_1 be one-loop quiver, that is, L_1 is a quiver with one vertex $\{1\}$ and 1-loop $\{\alpha\}$ such that $s(\alpha) = r(\alpha) = 1$. Consider two infinite-dimensional spaces the polynomial ring $\mathbb{C}[z]$ and the Hardy space $H^2(\mathbb{T})$. Then $\mathbb{C}[z]$ is dense in $H^2(\mathbb{T})$ with respect to the Hilbert space norm.

Define a purely algebraic representation (V, T) of L_1 by $V_1 = \mathbb{C}[z]$ and the multiplication operator T_α by z. That is, $T_\alpha h(z) = zh(z)$ for a polynomial $h(z) = \sum_n a_n z^n$. Since

$End(V,T) \cong \mathbb{C}[z]$ have no idempotents, the purely algebraic representation (V,T) is indecomposable.

Next we define a Hilbert representation (H,S) by $H_1 = H^2(\mathbb{T})$ and $S_\alpha = T_z$ the Toeplitz operator with the symbol z. Then $S_\alpha = T_z$ is the multiplication operator by z on $H^2(\mathbb{T})$ and is identified with the unilateral shift. Then

$$\begin{aligned} End(H,S) &\cong \{A \in B(H^2(\mathbb{T})) \mid AT_z = T_z A\} \\ &= \{T_\phi \in B(H^2(\mathbb{T})) \mid \phi \in H^\infty(\mathbb{T})\} \end{aligned}$$

is the algebra of analytic Toeplitz operators and isomorphic to $H^\infty(\mathbb{T})$. By the F. and M. Riesz Theorem, if $f \in H^2(\mathbb{T})$ has the zero set of positive measure, then $f = 0$. Since $H^\infty(\mathbb{T}) = H^2(\mathbb{T}) \cap L^\infty(\mathbb{T})$, $H^\infty(\mathbb{T})$ has no non-trivial idempotents. Thus there exists no non-trivial idempotents which commutes with T_z and Hilbert space (H,S) is indecomposable. In this sense, the analytical aspect of Hardy space is quite important in our setting.

Any subrepresentation of the purely algebraic representation (V,T) is given by the restriction to an ideal $J = p(z)\mathbb{C}[z]$ for some polynomial $p(z)$. Any subrepresentation of the Hilbert representation (H,S) is given by an invariant subspace of the shift operator T_z. Beurling theorem shows that any subrepresentation of (H,S) is given by the restriction to an invariant subspace $M = \varphi H^2(\mathbb{T})$ for some inner function φ. For example, if an ideal J is defined by

$$J = \{f(z) \in \mathbb{C}[z] \mid f(\lambda_1) = \cdots = f(\lambda_n) = 0\}$$

for some distinct numbers $\lambda_1, \ldots \lambda_n \in \mathbb{C}$, then the corresponding polynomial $p(z)$ is given by $p(z) = (z-\lambda_1)\ldots(z-\lambda_n)$. The case of Hardy space is much more analytic. We shall identify $H^2(\mathbb{T})$ with a subspace $H^2(\mathbb{D})$ of analytic functions on the open unit disc $\mathbb{D}$. If an invariant subspace M is defined by

$$M = \{f \in H^2(\mathbb{D}) \mid \ f(\lambda_1) = \cdots = f(\lambda_n) = 0\}$$

for some distinct numbers $\lambda_1, \ldots \lambda_n \in \mathbb{D}$, then the corresponding inner function φ is given by a finite Blaschke product

$$\varphi(z) = \frac{(z-\lambda_1)}{1-\overline{\lambda_1}z} \cdots \frac{(z-\lambda_n)}{1-\overline{\lambda_n}z}.$$

Here we cannot use the notion of degree like polynomials and we must manage to treat orthogonality to find such an inner function φ.

2. PATH ALGEBRAS AND CUNTZ-KRIEGER ALGEBRAS

The elements of a path algebra of a quiver are finite linear sums of paths in the quiver. Similarly the elements of a Cuntz-Krieger algebra of a quiver are generated by partial isometry operators representating paths. The difference is that the ranges of generating partial isometries are orthogonal and we add the adjoint T^* of any operator T in the Cuntz-Krieger algebra. But usually the Cuntz-Krieger algebra is described using the 0-1 matrix A corresponding the quiver as follows: The Cuntz-Krieger algebra $\mathcal{O}_A$ is the

symbolic dynamical system	complex dynamical system
quiver (or 0-1 matrix A)	rational function R
irreducible matrix	restriction of R on the Julia set J_R
Cantor set	closed subset of Riemann sphere
one-sided Markov shift	branched covering map
Cuntz-Krieger algebra $\mathcal{O}_A$	Cuntz-Pimsner algebra $\mathcal{O}_R(J_R)$
maximal abelian subalgebra $C(X_A)$	maximal abelian subalgebra $C(J_R)$
étale groupoid	not étale groupoid in general
K-group is a good invariant	K-group is not a good invariant

universal algebra generated by partial isometries $S_1, S_2, \ldots, S_n$ with orthogonal ranges satisfying that

$$S_i^* S_i = \sum_{j=1}^{n} A(i,j) S_j S_j^* \quad \text{and} \quad \sum_{j=1}^{n} S_j S_j^* = I$$

Theorem 1. (Cuntz-Krieger) Let A be an irreducible $0-1$ matrix which is not a permutation. Then the correspoding Cuntz-Krieger algebra $\mathcal{O}_A$ is simple, purely infinite, nuclear C^*-algebra. Furthermore the K-groups are the following:

$$K_0(\mathcal{O}_A) = \mathbb{Z}^n/(I - A^t)\mathbb{Z}^n \quad K_1(\mathcal{O}_A) = Ker\ (I - A^t) \subset \mathbb{Z}^n$$

It is known that the K-groups are complete invarint of a certain class of simple nuclear C^*-algebras containing Cuntz-Krieger algebras.

3. C^*-algebras associated with complex dynamical systems

We can regard Cuntz-Krieger algebras are C^*-algebraic version of path algebras for quivers. But we usually consider that Cuntz-Krieger algebras are associated with certain symbolic dynamical systems ,i.e. Markov shifts. Similarly many C^*-algebras are constructed from dynamical systems.

Let R be a rational function of the form $R(z) = \frac{P(z)}{Q(z)}$ with relatively prime polynomials P and Q. The degree of R is denoted by $N = \deg R := \max\{\deg P, \deg Q\}$.

We regard a rational function R as a N-fold branched covering map $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

The sequence $(R^n)_n$ of iterations of R gives a complex analytic dynamical system on $\hat{\mathbb{C}}$. The Fatou set F_R of R is the maximal open subset of $\hat{\mathbb{C}}$ on which $(R^n)_n$ is equicontinuous (or a normal family), and the Julia set J_R of R is the complement of the Fatou set in $\hat{\mathbb{C}}$. The Fatou set F_R is a stable part and the Julia set J_R is an unstable part. Since the Riemann sphere $\hat{\mathbb{C}}$ is decomposed into the union of the Julia set J_R and Fatou set F_R, we associated three C^*-algebras $\mathcal{O}_R = \mathcal{O}_R(\hat{\mathbb{C}})$, $\mathcal{O}_R(J_R)$ and $\mathcal{O}_R(F_R)$ by considering R as dynamical systems on $\hat{\mathbb{C}}$, J_R and F_R respectively.

Theorem 2. (Kajiwara-Watatani) Let R be a rational function with $\deg R \geq 2$. Then the C^*-algebra $\mathcal{O}_R(J_R)$ associated with R on the Julia set J_R is simple and purely infinite.

Theorem 3. (Kajiwara-Watatani) Let R be a rational function with $\deg R \geq 2$. Then the following are equivalent:
(1)The core $\mathcal{O}_R(J_R)^\gamma$ is simple.
(2)The Julia set J_R contains no branched points i.e. $J_R \cap B_R = \emptyset$.

4. Hilbert representation of quivers

Let $\Gamma = (V, E, s, r)$ be a finite quiver. We say that (H, f) is a *Hilbert representation* of Γ if $H = (H_v)_{v \in V}$ is a family of Hilbert spaces and $f = (f_\alpha)_{\alpha \in E}$ is a family of bounded linear operators such that $f_\alpha : H_{s(\alpha)} \to H_{r(\alpha)}$ for $\alpha \in E$.

Hilbert representation (H, f) of Γ is called *decomposable* if (H, f) is isomorphic to a direct sum of two non-zero Hilbert representations. A non-zero Hilbert representation (H, f) of Γ is said to be *indecomposable* if it is not decomposable, that is, if $(H, f) \cong (K, g) \oplus (K', g')$ then $(K, g) \cong 0$ or $(K', g') \cong 0$.

A Hilbert representation (H, f) of a quiver Γ is called *transitive* if $End(H, f) = \mathbb{C}I$. It is clear that if a Hilbert representation (H, f) is canonically simple, then (H, f) is transitive. If a Hilbert representation (H, f) of Γ is transitive , then (H, f) is indecomposable. In fact, since $End(H, f) = \mathbb{C}I$, any idempotent endomorphism T is 0 or I. In purely algebraic setting, a representation of a quiver is called a *brick* if its endomorphism ring is a division ring. But for a Hilbert representation (H, f) of a quiver, $End(H, f)$ is a Banach algebra and not isomorphic to its purely algebraic endomorphism ring in general, because we only consider bounded endomorphisms. By Gelfand-Mazur theorem, any Banach algebra over $\mathbb{C}$ which is a division ring must be isomorphic to $\mathbb{C}$. Therefore the reader may use "brick" instead of "transitive Hilbert representation" if he does not confuse the difference between purely algebraic endomorphism ring and $End(H, f)$.

A lattice $\mathcal{L}$ of subspaces of a Hilbert space H containing 0 and H is called a transitive lattice if

$$\{A \in B(H) \mid AM \subset M \text{ for any } M \in \mathcal{L}\} = \mathbb{C}I.$$

Let $\mathcal{L} = \{0, M_1, M_2, \ldots, M_n, H\}$ be a finite lattice. Consider a n subspace quiver $R_n = (V, E, s, r)$, that is, $V = \{1, 2, \ldots, n, n+1\}$ and $E = \{\alpha_k \mid k = 1, \ldots, n\}$ with $s(\alpha_k) = k$ and $r(\alpha_k) = n + 1$ for $k = 1, \ldots, n$. Then there exists a Hilbert representation (K, f) of R_n such that $K_k = M_k$, $K_{n+1} = H$ and $f_{\alpha_k} : M_k \to H$ is an inclusion for $k = 1, \ldots, n$. Then the lattice $\mathcal{L}$ is transitive if and only if the corresponding Hilbert representation (H, f) is transitive. This fact guarantees the terminology "transitive" in the above.

Theorem 4. (Enomoto-Watatani) Let Γ be a quiver whose underlying undirected graph is an extended Dynkin diagram. Then there exists an infinite-dimensional transitive Hilbert representation of Γ if and only if Γ is not an oriented cyclic quiver.

A non-zero Hilbert representation (H, f) of a quiver Γ is called *simple* if (H, f) has only trivial subrepresentations 0 and (H, f). A Hilbert representation (H, f) of Γ is called *canonically simple* if there exists a vertex $v_0 \in V$ such that $H_{v_0} = \mathbb{C}$, $H_v = 0$ for any other vertex $v \neq v_0$ and $f_\alpha = 0$ for any $\alpha \in E$. It is clear that if a Hilbert representation (H, f) of Γ is canonically simple, then (H, f) is simple. It is trivial that if a Hilbert representation (H, f) of Γ is simple, then (H, f) is indecomposable.

Operator theory	*Representation of quivers*
Hilbert space	vertex
bounded operator	edge
direct sum	direct sum
irreducible operator	irreducible representation
strongly irreducible operator T	indecomposable representation
commutant $\{T\}'$	endomorphism ring
up to similar	up to isomorphism
Fredholm index	defect
no non-trivial invariant subspace	simple
transitive operator	transitive representation

We can rephrase the invariant subspace problem in functional analysis in terms of simple representations of a one-loop quiver. Let L_1 be one-loop quiver, so that L_1 has one vertex 1 and one arrow $\alpha : 1 \to 1$. Any bounded operator A on a non-zero Hilbert space H gives a Hilbert representation (H^A, f^A) of L_1 such that $H_1^A = H$ and $f_\alpha^A = A$. Then the operator A has only trivial invariant subspaces if and only if the Hilbert representation (H^A, f^A) of L_1 is simple. If H is one-dimensional and A is a non-zero scalar operator, then the Hilbert representation (H^A, f^A) of L_1 is simple but is not canonically simple. If H is finite-dimensional with $\dim H \geq 2$, then the Hilbert representation (H^A, f^A) of L_1 is not simple, because any operator A on H has a non-trivial invariant subspace. If H is countably infinite-dimensional, then we do not know whether the Hilbert representation (H^A, f^A) of L_1 is not simple. In fact this is the invariant subspace problem, that is, the question whether any operator A on H has a non-trivial (closed) invariant subspace.

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Graduate School of Mathematics
Kyushu University
Motooka, Fukuoka, 819-0395 JAPAN
E-mail address: watatani@math.kyushu-u.ac.jp