ON WEAKLY SEPARABLE EXTENSIONS AND WEAKLY QUASI-SEPARABLE EXTENSIONS

SATOSHI YAMANAKA

Abstract. Separable extensions of noncommutative rings have already been studied extensively. Recently, N. Hamaguchi and A. Nakajima introduced the notions of weakly separable extensions and weakly quasi-separable extensions. The purpose of this paper is to give some improvements and generalizations of Hamaguchi and Nakajima's results. We shall characterize weakly separable polynomials, and we shall show the difference between the separability and the weakly separability in skew polynomial rings.

1. Introduction

Throughout this paper, A/B will represent a ring extension with common identity 1. Let M be an A-A-bimodule, and x, y arbitrary elements in A. An additive map δ is said to be a B-*derivation* of A to M if $\delta(xy) = \delta(x)y + x\delta(y)$ and $\delta(\alpha) = 0$ for any $\alpha \in B$. Moreover, δ is called *central* if $\delta(x)y = y\delta(x)$, and δ is called *inner* if $\delta(x) = mx - xm$ for some fixed element $m \in M$. We say that a ring extension A/B is *separable* if the A-A-homomorphism of $A \otimes_B A$ onto A defined by $a \otimes b \mapsto ab$ splits. It is well known that A/B is separable if and only if for any A-A-bimodule M, every B-derivation of A to M is inner (cf. [1, Satz 4.2]). In [12], Y. Nakai introduced the notion of a quasi-separable extension of commutative rings by using the module differentials, and in the noncommutative case, it was characterized by H. Komatsu [8, Lemma 2.1] as follows : A/B is *quasi-separable* if and only if for any A-A-bimodule M, every central B-derivation of A to M is zero. Recently, N. Hamaguchi and A. Nakajima gave the following definitions as generalizations of separable extensions and quasi-separable extensions.

Definition 1. ([2, Definition 2.1]) (1) A/B is called *weakly separable* if every B-derivation of A to A is inner.

(2) A/B is called *weakly quasi-separable* if every central B-derivation of A to A is zero.

Obviously, a separable extension is weakly separable and a quasi-separable extension is weakly quasi-separable. Moreover, a separable extension is quasi-separable by [8, Theorem 2.4].

Let B be a ring, ρ an automorphism of B, D a ρ-derivation, that is, D is an additive endomorphism of B such that $D(\alpha\beta) = D(\alpha)\rho(\beta) + \alpha D(\beta)$ for any $\alpha, \beta \in B$. $B[X; \rho, D]$ will mean the skew polynomial ring in which the multiplication is given by $\alpha X = X\rho(\alpha) + D(\alpha)$ for any $\alpha \in B$. We write $B[X; \rho] = B[X; \rho, 0]$ and $B[X; D] = B[X; 1, D]$. By $B[X; \rho, D]_{(0)}$ we denote the set of all monic polynomials g in $B[X; \rho, D]$ such that $gB[X; \rho, D] = B[X; \rho, D]g$. Let f be in $B[X; \rho, D]_{(0)}$. Then the residue ring $B[X; \rho, D]/fB[X; \rho, D]$ is a

The detailed version of this paper [15] will be published.

free ring extension of B. We say that f is a *separable* (resp. *weakly separable*, *weakly quasi-separable*) *polynomial* in $B[X;\rho,D]$ if $B[X;\rho,D]/fB[X;\rho,D]$ is separable (resp. weakly separable, weakly quasi-separable) over B.

In section 2, we treat weakly separable polynomials over a commutative ring B. In [10, Theorem 2.3], T. Nagahara showed that a separable polynomial $f(X)$ in $B[X]$ has a close relationship with the invertibilities of its derivative $f'(X)$ and its discriminant $\delta(f(X))$. We shall characterize the weakly separability of $f(X)$ in terms of the properties of $f'(X)$ and $\delta(f(X))$.

In section 3, we study weakly separable polynomials and weakly quasi-separable polynomials in skew polynomial rings. When B is an integral domain, N. Hamaguchi and A. Nakajima gave necessary and sufficient conditions for weakly separable polynomials in $B[X;\rho]$ and $B[X;D]$ (cf. [2, Theorem 4.1.4 and Theorem 4.2.3]). We shall try to give sharpenings of their results for a noncommutative coefficient ring B. Moreover, we shall show the difference between the separability and the weakly separability in skew polynomial rings $B[X;\rho]$ and $B[X;D]$, respectively.

2. Weakly separable polynomials over a commutative ring

In this section, we shall study weakly separable polynomials over a commutative ring. It is well known that a (noncommutative) ring extension A/B is separable if and only if there exists $\sum_j x_j \otimes y_j \in (A\otimes_B A)^A$ such that $\sum_j x_j y_j = 1$, where $(A\otimes_B A)^A = \{\mu \in A\otimes_B A \,|\, x\mu = \mu x$ for any $x \in A\}$ (cf. [3, Definition 2]). First we shall state the following.

Lemma 2. ([15, Lemma 2.1]) *Let A/B be a commutative ring extension. If there exists $\sum_j x_j \otimes y_j \in (A\otimes_B A)^A$ such that $\sum_j x_j y_j$ is a non-zero-divisor in A, then A/B is weakly separable.*

Now, let B be a commutative ring. For a monic polynomial $f(X) \in B[X]$, $f'(X)$ and $\delta(f(X))$ will mean the derivative of $f(X)$ and the discriminant of $f(X)$, respectively. As was shown in [10, Theorem 2.3], a polynomial $f(X)$ in $B[X]$ is separable if and only if $\delta(f(X))$ is invertible in B, or equivalently, $f'(X)$ is invertible in $B[X]$ modulo $(f(X))$. In [2], N. Hamaguchi and A. Nakajima proved that $f(X) = X^m - Xa - b$ is weakly separable in $B[X]$ if and only if $\delta(f(X))$ is a non-zero-divisor in B, or equivalently, $f'(X)$ is a non-zero-divisor in $B[X]$ modulo $(f(X))$ (cf [2, Theorem 3.1 and Corollary 3.2]). For a monic polynomial $f(X)$, we have the following.

Theorem 3. ([15, Theorem 2.2]) *Let B be a commutative ring, and $f(X)$ a monic polynomial in $B[X]$. The following are equivalent.*

(1) *$f(X)$ is weakly separable in $B[X]$.*
(2) *$f'(X)$ is a non-zero-divisor in $B[X]$ modulo $(f(X))$.*
(3) *$\delta(f(X))$ is a non-zero-divisor in B.*

Remark 4. In Theorem 3, it is already known that (2) and (3) are equivalent by [10, Theorem 1.3].

3. Weakly separable polynomials in skew polynomial rings

In [2], N. Hamaguchi and A. Nakajima characterized weakly separable polynomials and weakly quasi-separable polynomials in skew polynomial rings $B[X;\rho]$ and $B[X;D]$ when B is an integral domain. In this section, we shall generalize their results for a noncommutative coefficient ring B.

We shall use the following conventions :

$Z =$ the center of B
$V_A(B) =$ the centralizer of B in A
$B^\rho = \{\alpha \in B \mid \rho(\alpha) = \alpha\}$
$B^D = \{\alpha \in B \mid D(\alpha) = 0\}$ and $Z^D = Z \cap B^D$
$D(B) = \{D(\alpha) \mid \alpha \in B\}$

3.1. **Automorphism type.** We consider a polynomial f in $B[X;\rho]_{(0)}$ of the form

$$f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0 = \sum_{j=0}^{m} X^j a_j \quad (a_m = 1,\ m \geq 2).$$

By [4, Lemma 1.3], f is in $B[X;\rho]_{(0)}$ if and only if

$$\begin{cases} \alpha a_j = a_j \rho^{m-j}(\alpha) \ \ (\alpha \in B,\ 0 \leq j \leq m-1), \\ \rho(a_j) - a_j = a_{j+1}(\rho(a_{m-1}) - a_{m-1}) \ (0 \leq j \leq m-2), \\ a_0(\rho(a_{m-1}) - a_{m-1}) = 0. \end{cases}$$

We set $A = B[X;\rho]/fB[X;\rho]$, and $x = X + fB[X;\rho] \in A$. Now we let f be in $B[X;\rho]_{(0)} \cap B^\rho[X]$. Then there is an automorphism $\tilde{\rho}$ of A which is naturally induced by ρ, that is, $\tilde{\rho}$ is defined by $\tilde{\rho}(\sum_{j=0}^{m-1} x^j c_j) = \sum_{j=0}^{m-1} x^j \rho(c_j)$. We write $J_{\rho^k} = \{h \in A \mid \alpha h = h\rho^k(\alpha)\ (\alpha \in B)\}$ $(k \geq 1)$, $V = V_A(B)$, and $V^{\tilde{\rho}} = \{h \in V \mid \tilde{\rho}(h) = h\}$. Then we consider a $V^{\tilde{\rho}}$-$V^{\tilde{\rho}}$-homomorphism $\tau : J_\rho \longrightarrow J_{\rho^m}$ defined by

$$\begin{aligned} \tau(h) &= x^{m-1}\sum_{j=0}^{m-1} \tilde{\rho}^j(h) + x^{m-2}\sum_{j=0}^{m-2} \tilde{\rho}^j(h)a_{m-1} + \cdots + x\big\{\tilde{\rho}(h) + h\big\}a_2 + ha_1 \\ &= \sum_{k=0}^{m-1} x^k \sum_{j=0}^{k} \tilde{\rho}^j(h)a_{k+1}. \end{aligned}$$

First we shall state the following.

Lemma 5. ([15, Lemma 3.1]) *If δ is a B-derivation of A, then $\delta(x) \in J_\rho$ and $\tau(\delta(x)) = 0$. Conversely, if $g \in J_\rho$ with $\tau(g) = 0$, then there exists a B-derivation δ of A such that $\delta(x) = g$.*

Now we shall give the following which is a sharpening of [2, Theorem 4.1.4].

Theorem 6. ([15, Theorem 3.2]) *Let $f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0$ be in $B[X;\rho]_{(0)} \cap B^\rho[X]$. Then f is weakly separable in $B[X;\rho]$ if and only if*

$$\{g \in J_\rho \mid \tau(g) = 0\} = \{x(\tilde{\rho}(h) - h) \mid h \in V\}.$$

In virtue of Theorem 6, we shall show the difference between the separability and the weakly separability in $B[X;\rho]$ as follows.

Theorem 7. ([15, Theorem 3.4]) *Let m be the order of ρ, $f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0$ in $B[X;\rho]_{(0)} \cap B^\rho[X]$, $C(A)$ a center of A, and I_x an inner derivation of A by x (that is, $I_x(h) = hx - xh$ for any $h \in A$).*

(1) *f is weakly separable in $B[X;\rho]$ if and only if the following sequence of $V^{\tilde{\rho}}$-$V^{\tilde{\rho}}$-homomorphisms is exact:*

$$0 \longrightarrow C(A) \xrightarrow{\text{inj}} V \xrightarrow{I_x} J_\rho \xrightarrow{\tau} V^{\tilde{\rho}}.$$

(2) *f is separable in $B[X;\rho]$ if and only if the following sequence of $V^{\tilde{\rho}}$-$V^{\tilde{\rho}}$-homomorphisms is exact:*

$$0 \longrightarrow C(A) \xrightarrow{\text{inj}} V \xrightarrow{I_x} J_\rho \xrightarrow{\tau} V^{\tilde{\rho}} \longrightarrow 0.$$

Remark 8. In this section, we assumed that f is in $B[X;\rho]_{(0)} \cap B^\rho[X]$. However, in general case, a polynomial which is in $B[X;\rho]_{(0)}$ is not always in $B^\rho[X]$. Concerning this, we have already known by [4, Corollary 1.5] that if B is a semiprime ring then every polynomial in $B[X;\rho]_{(0)}$ is in $C(B^\rho)[X]$, where $C(B^\rho)$ is the center of B^ρ.

At the end of this section, we shall mention briefly on weakly quasi-separable polynomials in $B[X;\rho]$. When B is an integral domain, N. Hamaguchi and A. Nakajima proved that every polynomial in $B[X;\rho]_{(0)}$ is weakly quasi-separable (cf. [2, Theorem 4.1.1]). For an arbitrary ring B, we have the following.

Proposition 9. ([15, Proposition 3.5]) (1) *If $\rho \neq 1$ and $\{\rho(c) - c \,|\, c \in B\}$ contains a non-zero divisor, then every polynomial in $B[X;\rho]_{(0)}$ is weakly quasi-separable.*

(2) *Let $f = X^m - u$ be in $B[X;\rho]_{(0)}$. If m and u are non-zero-divisors in B, then f is weakly quasi-separable in $B[X;\rho]$.*

3.2. **Derivation type.** In this section, let B be of prime characteristic p, and we consider a p-polynomial f in $B[X;D]_{(0)}$ of the form

$$f = X^{p^e} + X^{p^{e-1}}b_e + \cdots + X^p b_2 + Xb_1 + b_0 = \sum_{j=0}^{e} X^{p^j} b_{j+1} + b_0 \ (b_{e+1} = 1).$$

We set $A = B[X;D]/fB[X;D]$, and $x = X + fB[X;D] \in A$. By [4, Corollary 1.7], f is in $B[X;D]_{(0)}$ if and only if

$$\begin{cases} b_0 \in B^D, \quad b_{j+1} \in Z^D \ (0 \leq j \leq e-1), \\ \displaystyle\sum_{j=0}^{e} D^{p^j}(\alpha) b_{j+1} = b_0\alpha - \alpha b_0 \quad (\alpha \in B). \end{cases}$$

Since f is in $B^D[X]$, there is a derivation $\tilde{D}$ of A which is naturally induced by D, that is, $\tilde{D}$ is defined by $\tilde{D}(\sum_{j=0}^{p^e-1} x^j c_j) = \sum_{j=0}^{p^e-1} x^j D(c_j)$. We write $V = V_A(B)$, $\tilde{D}(V) = \{\tilde{D}(h) \,|\, h \in$

$V\}$, and $V^{\tilde{D}} = \{v \in V \mid \tilde{D}(v) = 0\}$. Then we consider a $V^{\tilde{D}}$-$V^{\tilde{D}}$-homomorphism $\tau : V \longrightarrow V^{\tilde{D}}$ defined by

$$\begin{aligned}\tau(h) &= \tilde{D}^{p^e-1}(h) + \tilde{D}^{p^{e-1}-1}(h)b_e + \cdots + \tilde{D}^{p-1}(h)b_2 + hb_1 \\ &= \sum_{j=0}^{e} \tilde{D}^{p^j-1}(h)b_{j+1}.\end{aligned}$$

First we shall state the following.

Lemma 10. ([15, Lemma 3.7]) *If δ is a B-derivation of A, then $\delta(x) \in V$ and $\tau(\delta(x)) = 0$. Conversely, if $g \in V$ with $\tau(g) = 0$, then there exists a B-derivation δ of A such that $\delta(x) = g$.*

Now we shall give a sharpening of [2, Theorem 4.2.3]

Theorem 11. ([15, Theorem 3.8]) *Let $f = X^{p^e} + X^{p^{e-1}}b_e + \cdots + X^p b_2 + Xb_1 + b_0$ be in $B[X;D]_{(0)}$. Then f is weakly separable in $B[X;D]$ if and only if*

$$\{g \in V \mid \tau(g) = 0\} = \tilde{D}(V).$$

In virtue of Theorem 11, we shall show the difference between the separability and the weakly separability in $B[X;D]$ as follows.

Theorem 12. ([15, Theorem 3.10]) *Let $f = X^{p^e} + X^{p^{e-1}}b_e + \cdots + X^p b_2 + Xb_1 + b_0$ be in $B[X;D]_{(0)}$.*

(1) *f is weakly separable in $B[X;D]$ if and only if the following sequence of $V^{\tilde{D}}$-$V^{\tilde{D}}$-homomorphisms is exact:*

$$0 \longrightarrow V^{\tilde{D}} \xrightarrow{\text{inj}} V \xrightarrow{\tilde{D}} V \xrightarrow{\tau} V^{\tilde{D}}.$$

(2) *f is separable in $B[X;D]$ if and only if the following sequence of $V^{\tilde{D}}$-$V^{\tilde{D}}$-homomorphisms is exact:*

$$0 \longrightarrow V^{\tilde{D}} \xrightarrow{\text{inj}} V \xrightarrow{\tilde{D}} V \xrightarrow{\tau} V^{\tilde{D}} \longrightarrow 0.$$

Remark 13. Also when B is arbitrary ring and f is a monic polynomial in $B[X;D]_{(0)}$, Theorem 11 is true. However, we can not prove yet Theorem 12 in the case.

Finally, we shall mention briefly on weakly quasi-separable polynomials in $B[X;D]$. As same as automorphism type, N. Hamaguchi and A. Nakajima proved that every polynomial in $B[X;D]_{(0)}$ is weakly quasi-parable when B is an integral domain (cf. [2, Theorem 4.2.1]). For an arbitrary ring B, we have the following.

Proposition 14. ([15, Proposition 3.11]) (1) *If $D(B)$ contains a non-zero-divisor, then every polynomial in $B[X;D]_{(0)}$ is weakly quasi-seprable.*

(2) *Let $f = X^{p^e} + X^{p^{e-1}}b_e + \cdots + X^p b_2 + Xb_1 + b_0$ be in $B[X;D]_{(0)}$. If b_1 is a non-zero-divisor in B, then f is wakly quasi-separable.*

References

[1] S. Elliger, Über automorphismen und derivationen von ringen, *J. Reine Angew. Math.,* **277** 1975, 155–177.

[2] N. Hamaguchi and A. Nakajima, On generalizations of separable polynomials over rings, *Hokkaido Math. J.,* **42** 2013, no. 1, 53–68.

[3] K. Hirata and K. Sugano, On semisimple extensions and separable extensions over noncommutative rings, *J. Math. Soc. Japan* **18** (1966), 360–373.

[4] S. Ikehata, On separable polynomials and Frobenius polynomials in skew polynomial rings, *Math. J. Okayama Univ.,* **22** 1980, 115–129.

[5] S. Ikehata, Azumaya algebras and skew polynomial rings, *Math, J. Okayama Univ.,* **23** 1981, 19–32.

[6] S. Ikehata, On H-separable and Galois polynomials of degree p in skew polynomial rings, *Int. Math. Forum,* **3** 2008, no. 29-32, 1581-1586.

[7] K. Kishimoto, On abelian extensions of rings. I, *Math. J. Okayama Univ.,* **14** 1970, 159–174.

[8] H. Komatsu, Quasi-separable extensions of noncommutative rings, *Comm. Algebra,* **29** (3) 2001, 1011–1019.

[9] Y. Miyashita, On a skew polynomial ring, *J. Math. Soc. Japan,* **31** 1979, no. 2, 317–330.

[10] T. Nagahara, On separable polynomials of degree 2 in skew polynomial rings, *Math. J. Okayama Univ.,* **19** 1976, 65–95.

[11] T. Nagahara, On separable polynomials of degree 2 in skew polynomial rings II, *Math. J. Okayama Univ.,* **21** 1976, 167–177.

[12] Y. Nakai, On the theory of differentials in commutative rings, *J. Math. Soc. Japan,* **13** 1961, 63–84.

[13] S. Yamanaka and S. Ikehata, An alternative proof of Miyasita's theorem in a skew polynomial rings, *Int. J. Algebra,* **21** 2012, 1011–1023

[14] S. Yamanaka and S. Ikehata, On Galois polynomials of degree p in skew polynomial rings of derivation type, *Southeast Asian Bull. Math.,* **37** 2013, 625–634.

[15] S. Yamanaka, On weakly separable polynomials and weakly quasi-separable polynomials over rings, *Math. J. Okayama Univ.,* **58** 2016, 169–182.

Department of Mathematics
Graduate School of Natural Science and Technology
Okayama University
Okayama 700-8530 JAPAN

E-mail address: `s_yamanaka@math.okayama-u.ac.jp`