THE CLASSIFICATION OF TWO-TERM TILTING COMPLEXES FOR BRAUER GRAPH ALGEBRAS

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Abstract. The study of derived categories have been one of the central themes in representation theory. From Morita theoretic perspective, tilting complexes play an important role because the endomorphism algebras are derived equivalent to the original algebra [4]. It is well-known that derived equivalences preserve many homological properties. Thus it is important to classify tilting complexes for a given algebra. Our aim of this report is to give a classification of two-term tilting complexes for Brauer graph algebras.

1. Preliminaries

In this section, we collect some results which are necessary in this report. Throughout this report, K is an algebraically closed field. All algebras are assumed to be basic, indecomposable, and finite dimensional over K. We always work with finite dimensional right modules. For an algebra Λ, we denote by $\mathrm{mod}\Lambda$ the category of finite dimensional right Λ-modules and by $\mathrm{proj}\Lambda$ the full subcategory of $\mathrm{mod}\Lambda$ consisting of all finite dimensional projective Λ-modules. We sometimes write $\Lambda = KQ/I$, where Q is a quiver with relations I. We denote by P_i an indecomposable projective Λ-module corresponding to a vertex i of Q. An arrow of Q is identified to a map between indecomposable projective Λ-modules. The composition of maps $f : X \to Y$ and $g : Y \to Z$ is denoted as $gf : X \to Z$. For an object X, we denote by $|X|$ the number of isomorphism classes of indecomposable summands of X.

1.1. Tilting theory.

In this subsection, we recall the definition of tilting complexes. Let Λ be an algebra. We denote by $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)$ the bounded homotopy category of $\mathrm{proj}\Lambda$.

Definition 1. Let T be a complex in $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)$.

(1) We say that T is *pretilting* if $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)}(T, T[n]) = 0$ for all non-zero integers n.
(2) We say that T is *tilting* if it is pretilting and generates $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)$ by taking direct sums, direct summands, mapping cones and shifts.
(3) We say that T is *two-term* if it is of the form $(0 \to T^{-1} \to T^0 \to 0)$, where T^n is the n-th term of T.

We denote by $\mathsf{2\text{-}ptilt}\Lambda$ the set of isomorphism classes of indecomposable two-term pretilting complexes of Λ and by $\mathsf{2\text{-}tilt}\Lambda$ the set of isomorphism classes of basic two-term tilting complexes of Λ.

Proposition 2. [1, 3] *Let Λ be a symmetric algebra and T a two-term pretilting complex of Λ. Then the following hold:*

The detailed version of this paper will be submitted for publication elsewhere.

(1) *T satisfies* $\mathsf{add}T^0 \cap \mathsf{add}T^{-1} = 0$.
(2) *T is two-term tilting if and only if* $|T| = |\Lambda|$.

1.2. **Ribbon graphs and signed walks.** In this subsection, we introduce the notion of signed walks and admissible walks (see [2] for details). Throughout this report, we assume that all graphs contain no loops. A *ribbon graph* is a graph equipped with a cyclic ordering of the edges around each vertex. For a ribbon graph G, we denote by G_0 the set of vertices of G and by G_1 the set of edges of G. The degree $\mathrm{d}(v)$ of a vertex $v \in G_0$ is the number of edges incident to v.

Definition 3. A walk $w = (e_1, e_2, \ldots, e_l)$ (*i.e.*, it is a sequence of edges) of a graph is called a *signed walk* of a (ribbon) graph if it is equipped with a map $\epsilon : \{e_1, e_2, \ldots, e_l\} \to \{+1, -1\}$ such that $\epsilon(e_i) = -\epsilon(e_{i+1})$ for any $i \in \{1, 2, \ldots, l-1\}$. we call e_1, e_l the endpoints of the (signed) walk w. We often notate a signed walk by $(w; \epsilon)$ or $(e_1^{\epsilon(e_1)}, e_2^{\epsilon(e_2)}, \ldots, e_l^{\epsilon(e_l)})$. We denote by $\mathsf{SW}(G)$ the set of signed walks of a ribbon graph G.

To give a combinatorial description of an indecomposable two-term pretilting complex, we introduce a special signed walk, which is called an admissible walk.

Definition 4. We say that a signed walk $w = (e_1, \ldots, e_l; \epsilon)$ *satisfies the sign condition* if $\epsilon(e_1) = \epsilon(e_l)$ whenever the endpoints of w are same vertex. In general, two signed walks w and w' *satisfy the sign condition* if the signatures are same whenever two of four endpoints of w and w' are same vertex.

We will attach some extra data for a signed walk, which are uniquely determined by the signature. A *virtual edge* is an element in the set $\{\mathrm{vr}_-(e), \mathrm{vr}_+(e) \mid e \in G_1\}$. Let $(e_1, e_2, \ldots, e_k)_v$ be the cyclic ordering around a vertex $v \in G_0$. We define the cyclic ordering accounting the virtual edges as

$$(\mathrm{vr}_-(e_1), e_1, \mathrm{vr}_+(e_1), \mathrm{vr}_-(e_2), e_2, \mathrm{vr}_+(e_2), \ldots, \mathrm{vr}_-(e_k), e_k, \mathrm{vr}_+(e_k))_v.$$

For a signed walk $w = (e_1, \ldots, e_l; \epsilon)$, we define the following virtual edges attached to w:

$$e_0 := \mathrm{vr}_{-\epsilon(e_1)}(e_1), \quad e_{l+1} := \mathrm{vr}_{-\epsilon(e_l)}(e_l).$$

We also define $\epsilon(e_0) := -\epsilon(e_1)$ and $\epsilon(e_{l+1}) := -\epsilon(e_l)$. To improve readability of various statements, we only write down the edges required in the cyclic ordering around a vertex. For example, if the edges e, f, g are only important edges incident to a vertex v, then we will write the cyclic ordering $(e, f, g)_v$ instead of $(e, \ldots, f, \ldots, g, \ldots)_v$.

Let $w = (e_1, e_2, \ldots, e_n; \epsilon)$ and $w' = (e'_1, e'_2, \ldots, e'_m; \epsilon')$ be signed walks. Moreover, it is automatically understood what we mean by $e_0, e_{n+1}, e'_0, e'_{m+1}$ from the definition of virtual edges. Assume that a, b, c, d are edges incident to a vertex v given by

$$\{a, b\} := \{e_{i-1}, e_i\}, \quad \{c, d\} := \{e'_{j-1}, e'_j\}$$

for some $i \in \{1, 2, \ldots, n+1\}$ and $j \in \{1, 2, \ldots, m+1\}$. We say that v is an *intersecting vertex* of w and w' if a, b, c, d are pairwise distinct.

Definition 5. We say that w and w' is *non-crossing at the intersecting vertex* v if at most one of a, b, c, d is virtual, and the cyclic ordering around v with the signature is either

$$(a^+, b^-, c^+, d^-)_v \text{ or } (a^+, b^-, c^-, d^+)_v.$$

A *subwalk* of a walk w is consecutive subsequence of w. A *common walk* of two walks w and w' is a subwalk z of both w and w'. Moreover, it is said to be *maximal* if there is no common walk $z'(\neq z)$ of w and w' such that z is a subwalk of z'.

Definition 6. Let $w = (e_1, e_2, \dots, e_n; \epsilon)$ and $w' = (e'_1, e'_2, \dots, e'_m; \epsilon')$ be signed walks, and $z = (t_1, t_2, \dots, t_l)$ a maximal common subwalk of w and w'. Assume that u (respectively, v) is the endpoint of z for t_1 (respectively, t_l), and $t_k = e_{i+k-1} = e'_{j+k-1}$ for all $k \in \{1, 2, \dots, l\}$. We say that w and w' are *non-crossing at z* if the following hold:

- $\epsilon(t_k) = \epsilon'(t_k)$ for each $k \in \{1, 2, \dots, l\}$.
- With the exception of $i = j = 1$ and/or $m + 1 - i - l = n + 1 - j - l = 0$, the cyclic orderings around u and v are either

$$(t_1, e_{i-1}, e'_{j-1})_u \text{ and } (t_l, e'_{j+l}, e_{i+l})_v \text{ respectively,}$$
$$\text{or } (t_1, e'_{j-1}, e_{i-1})_u \text{ and } (t_l, e_{i+l}, e'_{j+l})_v \text{ respectively.}$$

We say that two signed walks w and w' are *non-crossing* if they are non-crossing at all maximal common subwalks and all intersecting vertices. In particular, w is *self-non-crossing* if w itself is non-crossing.

Definition 7. An admissible walk is a self-non-crossing signed walk which satisfies the sign condition. We denote by $\mathsf{AW}(G)$ the set of admissible walks of a ribbon graph G.

At the end of this subsection, we give the following result for finiteness of $\mathsf{AW}(G)$.

Proposition 8. [2, Proposition 2.12] *Let G be a ribbon graph. Then the following are equivalent:*

(1) $\mathsf{AW}(G)$ *is finite.*
(2) G *consists of at most one odd cycle and no even cycle.*

1.3. **Brauer graph algebras.** In this subsection, we recall the definition of Brauer graph algebras. A *Brauer graph* is a ribbon graph equipped with a map $\mathrm{m} : G_0 \to \mathbb{Z}_{>0}$, which is called multiplicity.

Let $G = (G, \mathrm{m})$ be a Brauer graph. Then we define the Brauer graph algebra Λ_G as follows: First, if G is the graph u —— v and $\mathrm{m}(u) = \mathrm{m}(v) = 1$, then $\Lambda_G = K[x]/(x^2)$. Otherwise, $\Lambda_G = KQ_G/I_G$, where

(1) Q_G is the following quiver:
  - There exists a one-to-one correspondence between the vertex of Q_G and the edges of G.
  - For two distinct vertices e and e' in Q_G corresponding to edges e and e' in G, we draw an arrow $\alpha_{e,e'} : e' \to e$ in Q_G if the edge e' is a direct successor of the edge e in the cyclic ordering around a common vertex in G. If the endpoint v of e in G satisfies $\mathrm{d}(v) = 1$ and $\mathrm{m}(v) > 1$, then we draw an arrow $\alpha_{e,e} : e \to e$ in Q_G.

(2) I_G is a two-sided ideal generated by the following relations: Let $(e_1, e_2, \dots, e_{\mathrm{d}(v)})_v$ be the cyclic ordering around $v \in G$. Then we define α_{e_j,e_i} to be the path

$$\alpha_{e_j,e_{j+1}} \cdots \alpha_{e_{i-2},e_{i-1}} \alpha_{e_{i-1},e_i}$$

in Q_G. Let $C_{e_i,v} := \alpha_{e_i,e_i}$.
  - If the edge e in G has endpoints u and v so that e is not a leaf at u with $\mathrm{m}(u) = 1$

or at v with $\mathrm{m}(v) = 1$, then $C_{e,u}^{\mathrm{m}(u)} - C_{e,v}^{\mathrm{m}(v)} \in I_G$.

• If the edge e in G has endpoints u and v so that e is a leaf at u with $\mathrm{m}(u) = 1$, then $C_{e,v}^{\mathrm{m}(v)}\alpha_{e,e'} \in I_G$, where e' is a direct predecessor of e in the cyclic ordering.

• All path $\alpha\beta$ which is not a subpath of any cycle $C_{e,v}$ are in I_G.

It is well-known that each Brauer graph algebra is a symmetric special biserial algebra, and vice versa [5]. In particular, an indecomposable non-projective module is either a string module or a band module [6]. Note that, for an indecomposable two-term complex T, if the 0-th cohomology $\mathrm{H}^0(T)$ is band, then T is not pretilting. Hence we are interested in only string modules in this report.

2. Main results

Let $G = (G, \mathrm{m})$ be a Brauer graph and $\Lambda = \Lambda_G$ the Brauer graph algebra.

Definition 9. An indecomposable two-term complex T is called a *string complex* if the 0-th cohomology $\mathrm{H}^0(T)$ is a string module. We denote by $\mathsf{2\text{-}scx}\Lambda$ the set of indecomposable stalk complexes of projective modules concentrated in degree 0 or -1, and string complexes $T = (T^{-1} \to T^0)$ with $\mathsf{add}T^0 \cap \mathsf{add}T^{-1} = 0$.

Lemma 10. [2, Lemma 4.4] $\mathsf{2\text{-}ptilt}\Lambda$ *is a subset of* $\mathsf{2\text{-}scx}\Lambda$.

For a signed walk $w = (e_1, e_2, \dots, e_n; \epsilon)$, we define a two-term complex $T_w = (T^{-1} \xrightarrow{d} T^0)$ as follows:

• $T^0 := \bigoplus_{\epsilon(e_i)=+1} P_{e_i}$ and $T^{-1} := \bigoplus_{\epsilon(e_i)=-1} P_{e_i}$.

• $d = (d_{ij})$, where $d_{ij} : P_{e_j} \to P_{e_i}$ given by

$$d_{ij} := \begin{cases} \alpha_{e_i,e_j} & (|i-j| = 1) \\ 0 & (\text{otherwise}) \end{cases}$$

Note that T_w is in $\mathsf{2\text{-}scx}\Lambda$. On the other hand, for a two-term complex $T \in \mathsf{2\text{-}scx}\Lambda$, we can easily construct a signed walk w_T because $\mathrm{H}^0(T)$ is string. The following proposition plays important role in this report.

Proposition 11. [2, Lemma 4.3] *There are mutually inverse bijections*

$$\mathsf{SW}(G) \longleftrightarrow \mathsf{2\text{-}scx}\Lambda$$

given by $w \mapsto T_w$ *and* $T \to w_T$. *Moreover, the restrictions give mutually inverse bijections*

$$\mathsf{AW}(G) \longleftrightarrow \mathsf{2\text{-}ptilt}\Lambda.$$

Using the correspondences, we state our main result. A collection of admissible walks is *admissible* if any pair in the collection is non-crossing and satisfies the sign condition. Moreover, an admissible collection $\mathbb{W}$ called *complete* if any admissible collection containing $\mathbb{W}$ is $\mathbb{W}$ itself. We denote by $\mathsf{CW}(G)$ the set of all complete admissible collections of G.

Theorem 12. [2, Theorem 4.6] *The correspondences in Proposition 11 induce bijections*

$$\mathsf{CW}(G) \longleftrightarrow \mathsf{2\text{-}tilt}\Lambda.$$

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