THE GROTHENDIECK GROUPS OF MESH ALGEBRAS

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Abstract. This note is written on the Grothendieck groups of the stable categories of finite-dimensional mesh algebras.

1. Introduction

This note is the collection of the results of our calculations of the Grothendieck groups of the stable categories of finite-dimensional mesh algebras.

The concepts of mesh algebras and mesh categories are proposed by Riedtmann, and important because many derived categories are recovered from the mesh categories of their Auslander-Reiten quivers. For example, if Γ is the path algebra of a quiver with its underlying graph a Dynkin diagram Δ, then $D^{\mathrm{b}}(\operatorname{mod}\Gamma)$ is recovered from the mesh category of its AR quiver $\boldsymbol{Z}\Delta$ [3].

Some of the results in this note have been obtained in [1], but this note is based on different methods from the ones in [1]. The detail of the new methods and the calculations will be submitted later.

1.1. **Conventions.** In this note, let K be a field and Λ be a finite-dimensional self-injective K-algebra. $\operatorname{mod}\Lambda$ denotes the category of finitely generated right Λ-modules. $\operatorname{proj}\Lambda$ is the fullsubcategory of $\operatorname{mod}\Lambda$ consisting of all projective Λ-modules, and $\underline{\operatorname{mod}}\Lambda = \operatorname{mod}\Lambda/\operatorname{proj}\Lambda$ is the *stable category* of $\operatorname{mod}\Lambda$. Because Λ is self-injective, $\operatorname{mod}\Lambda$ is an abelian Frobenius category and $\underline{\operatorname{mod}}\Lambda$ has a structure of a triangulated category [3]. The unit 1_Λ is decomposed into primitive orthogonal idempotents $e_1 + \cdots + e_m$. In this case, we put $P_i = e_i\Lambda$, $I_i = \operatorname{Hom}_K(\Lambda e_i, K)$, and $S_i = \operatorname{top} P_i = \operatorname{soc} I_i$. We define *Nakayama permutation* ν as $P_i \cong I_{\nu(i)}$.

2. Preliminary

First, we recall basic properties on Grothendieck groups and mesh algebras. We can refer to [3] for the detail.

Definition 1. Let $\mathcal{C}$ be a triangulated category.

The *Grothendieck group* $K_0(\mathcal{C})$ is defined with its generators all isomorphic classes in $\mathcal{C}$ and its relations $[X] - [Y] + [Z] = 0$ for each triangle $X \to Y \to Z \to X[1]$.

We have the following important proposition to calculate the Grothendieck group of the stable category $\underline{\operatorname{mod}}\Lambda$. The latter part of (2) is deduced by Rickard's famous triangle equivalence $\underline{\operatorname{mod}}\Lambda \cong D^{\mathrm{b}}(\operatorname{mod}\Lambda)/K^{\mathrm{b}}(\operatorname{proj}\Lambda)$ [4, Theorem 2.1].

Proposition 2. *Let Λ be a finite-dimensional self-injective K-algebra.*

The detailed version of this paper will be submitted for publication elsewhere.

(1) [3, III.1.2] *All isomorphic classes of simple Λ-modules $[S_1], \dots, [S_m]$ form a basis of the Grothendieck group of the derived category $K_0(D^{\mathrm{b}}(\operatorname{mod} \Lambda))$.*
(2) *The natural embedding $K^{\mathrm{b}}(\operatorname{proj} \Lambda) \to D^{\mathrm{b}}(\operatorname{mod} \Lambda)$ canonically induces a morphism $K_0(K^{\mathrm{b}}(\operatorname{proj} \Lambda)) \to K_0(D^{\mathrm{b}}(\operatorname{mod} \Lambda))$, and its cokernel is isomorphic to $K_0(\underline{\operatorname{mod}} \Lambda)$.*

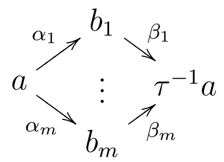
Definition 3. Let $Q = (Q_0, Q_1)$ be a locally finite quiver and τ be an automorphism on Q_0. We call the pair $Q = (Q, \tau)$ a *stable translation quiver* if the number of arrows from x to y coincide with the one from y to $\tau^{-1}x$ for $x, y \in Q_0$.

It will be seen that a stable translation quiver with multiple arrows does not give a finite-dimensional mesh algebra from Rickard's structure theorem. Thus, in this note, we assume any stable translation quivers do not contain multiple arrows for the convinience.

Definition 4. Let Q be a stable translation quiver.

For a vertex $a \in Q_0$, we denote by a^+ the set of targets of arrows from a^+.

Let $b_1, \dots, b_m$ be all distinct elements of a^+. Then the full subquiver



is called a *mesh* and the corresponding *mesh relation* is $\alpha_1\beta_1 + \dots + \alpha_m\beta_m = 0$.

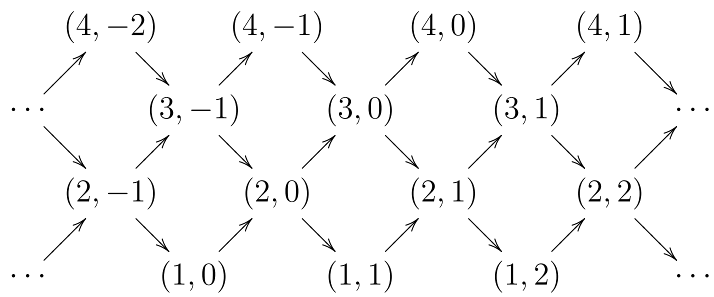
We define the *mesh algebra* of Q as the quotient of the path algebra of Q by all mesh relations in Q.

The following example introduces an important way to construct a translation quiver.

Example 5. Let Q be a finite quiver. We define the quiver $\boldsymbol{Z}Q = ((\boldsymbol{Z}Q)_0, (\boldsymbol{Z}Q)_1)$ as follows; the vertices are the elements of $(\boldsymbol{Z}Q)_0 = Q_0 \times \boldsymbol{Z}$, the arrows are the elements of $(\boldsymbol{Z}Q)_1 = \{(i, a) \to (j, a) \mid (i \to j) \in Q_1, a \in \boldsymbol{Z}\} \amalg \{(j, a) \to (i, a+1) \mid (i \to j) \in Q_1, a \in \boldsymbol{Z}\}$, and the translation is given by $\tau(i, a) = (i, a-1)$. Then $\boldsymbol{Z}Q$ is a stable translation quiver.

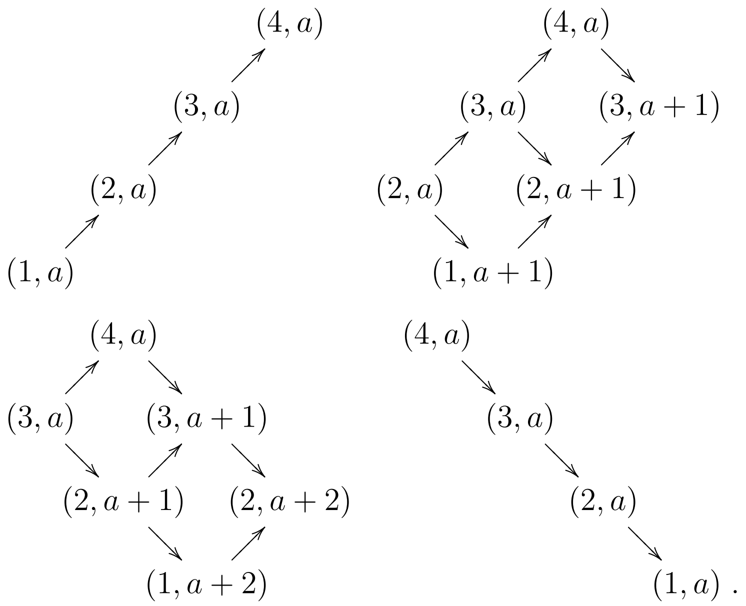
Remark 6. If the underlying graph of Q is a Dynkin diagram Δ, the translation quiver $\boldsymbol{Z}Q$ does not depend on the orientations of Q up to isomorphism, thus we set $\boldsymbol{Z}\Delta = \boldsymbol{Z}Q$.

Example 7. Let A_4 be oriented as $1 \to 2 \to 3 \to 4$. Then $\boldsymbol{Z}A_4$ is the following quiver



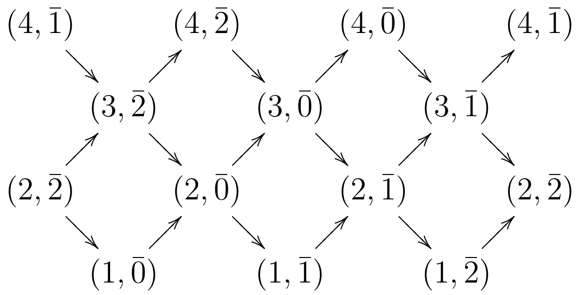
with its translation $\tau(i, a) = (i, a-1)$.

Considering mesh relations, the following paths are the longest nonzero paths in $\boldsymbol{Z}A_4$;



The second figure means all paths from $(2, a)$ to $(3, a + 1)$ are the longest nonzero paths. We can see that any of the longest nonzero paths from (i, a) ends at $(5 - i, a + i - 1)$.

To get a finite-dimensional mesh algebra, we take the quotient of $\boldsymbol{Z}A_4$ by an automorphism τ^3. Then the quiver $\boldsymbol{Z}A_4/\langle\tau^3\rangle$ is the following quiver



with its relation $\tau(i, a + k\boldsymbol{Z}) = (i, a - 1 + k\boldsymbol{Z})$. This quiver looks like a cylinder. From the discussion on the longest nonzero paths, we have $\nu(i, a+k\boldsymbol{Z}) = (5-i, a+i-1+k\boldsymbol{Z})$.

We can deduce the following lemma similarly as above.

Lemma 8. *A translation quiver $\boldsymbol{Z}A_n/\langle\tau^k\rangle$ gives a finite-dimensional mesh algebra for integers $n, k \geq 1$. The Nakayama permutation of this mesh algebra is given by $\nu(i, a + k\boldsymbol{Z}) = (n + 1 - i, a + i - 1 + k\boldsymbol{Z})$.*

Actually, it is rare for mesh algebras to be finite-dimensional. This is stated in Riedtmann's structure theorem.

Theorem 9. [5] *If a stable translation quiver gives a finite-dimensional mesh algebra, then it has a form of $\boldsymbol{Z}\Delta/G$, where Δ is a Dynkin diagram, and G is an admissible subgroup of* Aut $\boldsymbol{Z}\Delta$*. Namely, it is isomorphic to one of the following translation quivers;*

$\boldsymbol{Z}A_n/\langle\tau^k\rangle$, $\boldsymbol{Z}A_n/\langle\tau^k\psi\rangle$ *with* n *odd,* $\boldsymbol{Z}A_n/\langle\tau^k\varphi\rangle$ *with* n *even,*
$\boldsymbol{Z}D_n/\langle\tau^k\rangle$, $\boldsymbol{Z}D_n/\langle\tau^k\psi\rangle$, $\boldsymbol{Z}D_4/\langle\tau^k\chi\rangle$,
$\boldsymbol{Z}E_6/\langle\tau^k\rangle$, $\boldsymbol{Z}E_6/\langle\tau^k\psi\rangle$, $\boldsymbol{Z}E_7/\langle\tau^k\rangle$, $\boldsymbol{Z}E_8/\langle\tau^k\rangle$;
where ψ, χ, φ *are automorphisms on* $\boldsymbol{Z}\Delta$ *satisfying* $\psi^2 = \mathrm{id}$, $\chi^3 = \mathrm{id}$, *and* $\varphi^2 = \tau^{-1}$.

It is well-known that all finite-dimensional mesh algebras are self-injective.

3. Results

In the previous section, all finite-dimensional mesh algebras are obtained. We can state the following main theorem on the Grothendieck groups of finite-dimensional mesh algebras. This is the collection of our main results.

Theorem 10. *Let* $Q = \boldsymbol{Z}\Delta/G$ *be a stable translation quiver giving a finite-dimensional mesh algebra* Λ. *Then the Grothendieck group* $K_0(\underline{\mathrm{mod}}\,\Lambda)$ *is isomorphic to the following, where* c *be the Coxeter number of* Δ,

$$d = \begin{cases} \gcd(c, 2k-1)/2 & (\boldsymbol{Z}\Delta/G = \boldsymbol{Z}A_n/\langle\tau^k\varphi\rangle) \\ \gcd(c, k) & (\text{otherwise}) \end{cases}$$

and $r = c/d$;

$$Q = \boldsymbol{Z}A_n/\langle\tau^k\rangle \Rightarrow \begin{cases} \boldsymbol{Z}^{(nd-3d+2)/2} \oplus (\boldsymbol{Z}/2\boldsymbol{Z})^{d-1} & (r \in 2\boldsymbol{Z}) \\ \boldsymbol{Z}^{(nd-2d+2)/2} & (r \notin 2\boldsymbol{Z}) \end{cases},$$

$$\begin{matrix} Q = \boldsymbol{Z}A_n/\langle\tau^k\psi\rangle \\ (n \notin 2\boldsymbol{Z}) \end{matrix} \Rightarrow \begin{cases} \boldsymbol{Z}^{(nd-3d)/2} \oplus (\boldsymbol{Z}/2\boldsymbol{Z})^{d-1} \oplus (\boldsymbol{Z}/4\boldsymbol{Z}) & (r \in 4\boldsymbol{Z}) \\ (\boldsymbol{Z}/2\boldsymbol{Z})^{nd-2d+1} & (r \in 2+4\boldsymbol{Z}) \\ \boldsymbol{Z}^{(nd-d)/4} & (r \notin 2\boldsymbol{Z}) \end{cases},$$

$$\begin{matrix} Q = \boldsymbol{Z}A_n/\langle\tau^k\varphi\rangle \\ (n \in 2\boldsymbol{Z}) \end{matrix} \Rightarrow (\boldsymbol{Z}/2\boldsymbol{Z})^{nd-2d+1},$$

$$Q = \boldsymbol{Z}D_n/\langle\tau^k\rangle \Rightarrow \begin{cases} \boldsymbol{Z}^{d-1} \oplus (\boldsymbol{Z}/2\boldsymbol{Z})^{nd-3d} \oplus (\boldsymbol{Z}/r\boldsymbol{Z}) & (k \in 2\boldsymbol{Z},\ r \in 2\boldsymbol{Z}) \\ \boldsymbol{Z}^{(nd-d-2)/2} \oplus (\boldsymbol{Z}/r\boldsymbol{Z}) & (k \in 2\boldsymbol{Z},\ r \notin 2\boldsymbol{Z}) \\ \boldsymbol{Z}^{d} \oplus (\boldsymbol{Z}/2\boldsymbol{Z})^{nd-3d} & (k \notin 2\boldsymbol{Z},\ r \in 4\boldsymbol{Z}) \\ (\boldsymbol{Z}/2\boldsymbol{Z})^{nd-d-1} & (k \notin 2\boldsymbol{Z},\ r \notin 4\boldsymbol{Z}) \end{cases},$$

$$Q = \boldsymbol{Z}D_n/\langle\tau^k\psi\rangle \Rightarrow \begin{cases} \boldsymbol{Z}^{d} \oplus (\boldsymbol{Z}/2\boldsymbol{Z})^{nd-3d} & (k \in 2\boldsymbol{Z},\ r \in 4\boldsymbol{Z}) \\ (\boldsymbol{Z}/2\boldsymbol{Z})^{nd-d-1} & (k \in 2\boldsymbol{Z},\ r \in 2+4\boldsymbol{Z}) \\ \boldsymbol{Z}^{(nd-2d)/2} & (k \in 2\boldsymbol{Z},\ r \notin 2\boldsymbol{Z}) \\ \boldsymbol{Z}^{d-1} \oplus (\boldsymbol{Z}/2\boldsymbol{Z})^{nd-3d} \oplus (\boldsymbol{Z}/r\boldsymbol{Z}) & (k \notin 2\boldsymbol{Z}) \end{cases},$$

$$Q = \boldsymbol{Z}D_4/\langle\tau^k\chi\rangle \Rightarrow \begin{cases} \boldsymbol{Z}^{4} & (k \in 2\boldsymbol{Z}) \\ (\boldsymbol{Z}/2\boldsymbol{Z})^{4} & (k \notin 2\boldsymbol{Z}) \end{cases},$$

$$Q = \boldsymbol{Z}E_6/\langle\tau^k\rangle \Rightarrow \begin{cases} \boldsymbol{Z}^{d+1} \oplus (\boldsymbol{Z}/2\boldsymbol{Z})^{d+1} \oplus (\boldsymbol{Z}/4\boldsymbol{Z})^{d-1} & (d = 1, 3) \\ \boldsymbol{Z}^{(3d+2)/2} \oplus (\boldsymbol{Z}/2\boldsymbol{Z})^{(3d+2)/2} & (d = 2, 6) \\ \boldsymbol{Z}^{(9d+12)/4} & (d = 4, 12) \end{cases},$$

$$Q = \boldsymbol{Z}E_6/\langle\tau^k\psi\rangle \Rightarrow \begin{cases} \boldsymbol{Z}^{2d} \oplus (\boldsymbol{Z}/2\boldsymbol{Z})^{d+1} & (d=1,3) \\ (\boldsymbol{Z}/2\boldsymbol{Z})^{(9d+6)/2} & (d=2,6) \\ \boldsymbol{Z}^{(3d+4)/2} & (d=4,12) \end{cases},$$

$$Q = \boldsymbol{Z}E_7/\langle\tau^k\rangle \Rightarrow \begin{cases} (\boldsymbol{Z}/2\boldsymbol{Z})^{6} & (d=1) \\ (\boldsymbol{Z}/2\boldsymbol{Z})^{6d+2} & (d=3,9) \\ \boldsymbol{Z}^{6} \oplus (\boldsymbol{Z}/3\boldsymbol{Z}) & (d=2) \\ \boldsymbol{Z}^{3d+2} & (d=6,18) \end{cases},$$

$$Q = \boldsymbol{Z}E_8/\langle\tau^k\rangle \Rightarrow \begin{cases} (\boldsymbol{Z}/2\boldsymbol{Z})^{8d} & (d=1,3,5) \\ (\boldsymbol{Z}/2\boldsymbol{Z})^{112} & (d=15) \\ \boldsymbol{Z}^{4d} & (d=2,6,10) \\ \boldsymbol{Z}^{112} & (d=30) \end{cases}.$$

4. Proof for $\boldsymbol{Z}A_n/\langle\tau^k\rangle$

In the rest of this note, we prove the main theorem for $\boldsymbol{Z}A_n/\langle\tau^k\rangle$. We orient A_n as $1 \to 2 \to \cdots \to n$, and set $Q = \boldsymbol{Z}A_n/\langle\tau^k\rangle$. The vertices of Q are the elements of $\{1,\ldots,n\} \times (\boldsymbol{Z}/k\boldsymbol{Z})$. The following proposition is crucial to prove the theorem.

Proposition 11. *Let three abelian subgroups $H, H', H'' \subset K_0(D^{\mathrm{b}}(\mathrm{mod}\,\Lambda))$ be*

$$H = \langle [P_x] \mid x \in Q_0\rangle, \quad H' = \langle [S_x] + [S_{\nu\tau^{-1}x}] \mid x \in Q_0\rangle,$$
$$H'' = \langle [P_x] \mid x \in \{1\} \times (\boldsymbol{Z}/k\boldsymbol{Z})\rangle \subset H.$$

Then we have $H = H' + H''$ and thus $K_0(\underline{\mathrm{mod}}\,\Lambda) \cong K_0(D^{\mathrm{b}}(\mathrm{mod}\,\Lambda))/(H' + H'')$.

Proof. Let $x \in Q_0$. A projective resolution of Λ-module S_x has a form of

$$0 \to S_{\nu\tau^{-1}x} \to P_{\tau^{-1}x} \to \bigoplus_{y \in x^+} P_y \to P_x \to S_x \to 0.$$

This is induced by a projective resolution of Λ as Λ-Λ-bimodule given by [2, (4.1)–(4.3), Corollary 4.3].

Now we prove $H' + H'' \subset H$. $H'' \subset H$ is clear. $H' \subset H$ holds because the above projective resolution implies

$$[S_x] + [S_{\nu\tau^{-1}x}] = [P_{\tau^{-1}x}] - \sum_{y \in x^+} [P_y] + [P_x] \in H.$$

We have $H' + H'' \subset H$.

The remained task is to prove $H \subset H' + H''$. We assume $k = 1$ and $Q_0 = \{1,\ldots,n\}$ first. It is enough to show $[P_i] \in H' + H''$. We prove this by induction on i. If $i = 1$, then $[P_1] \in H''$. If $i = 2,\ldots,n$, put $x = i-1$. The projective resolution of Λ-module S_x implies

$$\begin{aligned}[S_x] + [S_{\nu\tau^{-1}x}] &= [P_{\tau^{-1}x}] - \sum_{y \in x^+} [P_y] + [P_x] \\ &= [P_{i-1}] - ([P_{i-2}] + [P_i]) + [P_{i-1}],\end{aligned}$$

where we set $P_0 = 0$. Therefore, we have

$$[P_i] = -([S_x] + [S_{\nu\tau^{-1}x}]) + [P_{i-1}] - [P_{i-2}] + [P_{i-1}].$$

From the induction hypothesis, we have $[P_{i-1}]-[P_{i-2}]+[P_{i-1}] \in H'+H''$, and by definition, we have $[S_x] + [S_{\nu\tau^{-1}x}] \in H'$. Now, $[P_i] \in H' + H''$ is proved. The induction has been completed. A similar proof holds even if $k \neq 1$. We have $H = H' + H''$.

The latter assertion is proved by Proposition 2. □

Now our task is moved to express the generators of H' and H'' as linear combinations of the images of simple Λ-modules. For this purpose, we define some matrices.

Definition 12. We define three matrices.

(1) $X_k \in \mathrm{GL}_k(\boldsymbol{Z})$ as the permutation matrix of a cyclic permutation $(1, 2, \ldots, k)$.
(2) $T_n(x) \in \mathrm{Mat}_{n,n}(\boldsymbol{Z}[x])$, $U_n(x) \in \mathrm{Mat}_{n,1}(\boldsymbol{Z}[x])$ as

$$T_n(x) = \begin{pmatrix} & & & x^n \\ & & \cdots & \\ & x^2 & & \\ x & & & \end{pmatrix}, \quad U_n(x) = \begin{pmatrix} 1 \\ 1 \\ \cdots \\ 1 \end{pmatrix}.$$

For example,

$$X_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Using these matrices, the Grothendieck group is written in the following way.

Lemma 13. *We have* $K_0(\underline{\mathrm{mod}}\,\Lambda) \cong \mathrm{Cok}\begin{pmatrix} 1_{nk} + T_n(X_k) & U_n(X_k) \end{pmatrix}$.

Proof. For $i \in \{1, \ldots, n\}$ and $a \in \{0, \ldots, k-1\}$, we let the $(i-1)k + (a+1)$th row of the matrix in the right-hand side correspond to $[S_{i,a+k\boldsymbol{Z}}]$, the element of the basis of $K_0(D^{\mathrm{b}}(\mathrm{mod}\,\Lambda))$. Then it is easy to see the columns of $1_{nk} + T_n(X_k)$ and $U_n(X_k)$ correspond to the generators of H' and H'', respectively. Using Proposition 11, we have the assertion. □

We consider transformations of $\begin{pmatrix} 1_n + T_n(x) & U_n(x) \end{pmatrix}$ in $\mathrm{Mat}_{n+1,n}(\boldsymbol{Z}[x])$.

Example 14. If $n = 7$, $\begin{pmatrix} 1 + T_n(x) & U_n(x) \end{pmatrix}$ is

$$\begin{pmatrix} 1 & & & & & & x^7 & 1 \\ & 1 & & & & x^6 & & 1 \\ & & 1 & & x^5 & & & 1 \\ & & & 1+x^4 & & & & 1 \\ & & x^3 & & 1 & & & 1 \\ & x^2 & & & & 1 & & 1 \\ x & & & & & & 1 & 1 \end{pmatrix}.$$

This can be transformed as a matrix on $\boldsymbol{Z}[x]$ as follows;

$$\begin{pmatrix}
1 &&&&&&& \\
& 1 &&&&&& \\
&& 1 &&&&& \\
&&& 1+x^4 &&&& 1 \\
&&&& 1-x^8 &&& 1-x^3 \\
&&&&& 1-x^8 && 1-x^2 \\
&&&&&& 1-x^8 & 1-x
\end{pmatrix}$$

$$\mapsto \begin{pmatrix}
1 &&&&&&& \\
& 1 &&&&&& \\
&& 1 &&&&& \\
&&& 1+x^4 &&&& 1 \\
&&&& 1-x^8 &&& \\
&&&&& 1-x^8 && \\
&&&&&& 1-x^8 & 1-x
\end{pmatrix}$$

$$\mapsto \begin{pmatrix}
1 &&&&&&& \\
& 1 &&&&&& \\
&& 1 &&&&& \\
&&& 0 &&&& 1 \\
&&&& 1-x^8 &&& \\
&&&&& 1-x^8 && \\
&&& -(1-x)(1+x^4) &&& 1-x^8 & 0
\end{pmatrix}$$

$$\mapsto \begin{pmatrix}
1 &&&&&&& \\
& 1 &&&&&& \\
&& 1 &&&&& \\
&&& 0 &&&& 1 \\
&&&& 1-x^8 &&& \\
&&&&& 1-x^8 && \\
&&& -(1-x)(1+x^4) &&& 0 & 0
\end{pmatrix}.$$

Thus we have $\operatorname{Cok}\big(1_{7k} + T_7(X_k) \quad U_7(X_k)\big) \cong (\operatorname{Cok}(1 - X_k^8))^2 \oplus \operatorname{Cok}((1 - X_k)(1 + X_k^4))$.

If $n = 6$, $\big(1 + T_n(x) \quad U_n(x)\big)$ is

$$\begin{pmatrix}
1 &&&&& x^6 & 1 \\
& 1 &&& x^5 && 1 \\
&& 1 & x^4 &&& 1 \\
&& x^3 & 1 &&& 1 \\
& x^2 &&& 1 && 1 \\
x &&&&& 1 & 1
\end{pmatrix}.$$

This can be transformed as a matrix on $\boldsymbol{Z}[x]$ as follows;

$$\begin{pmatrix}
1 &&&&&& \\
& 1 &&&&& \\
&& 1 &&&& \\
&&& 1-x^7 &&& 1-x^3 \\
&&&& 1-x^7 && 1-x^2 \\
&&&&& 1-x^7 & 1-x
\end{pmatrix}$$

$$\mapsto \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1-x^7 & & & \\ & & & & 1-x^7 & & \\ & & & & & 1-x^7 & 1-x \end{pmatrix}$$

$$\mapsto \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1-x^7 & & & \\ & & & & 1-x^7 & & \\ & & & & & 0 & 1-x \end{pmatrix}.$$

Thus we have $\mathrm{Cok}\left(1_{6k} + T_6(X_k) \quad U_6(X_k)\right) \cong (\mathrm{Cok}(1 - X_k^7))^2 \oplus \mathrm{Cok}(1 - X_k)$.

These examples are generalized as follows.

Lemma 15. $K_0(\underline{\mathrm{mod}}\,\Lambda) \cong \mathrm{Cok}\left(1 + T_n(X_k) \quad U_n(X_k)\right)$ *is isomorphic to*

$$\begin{cases} (\mathrm{Cok}(1_k - X_k^{n+1}))^{(n-3)/2} \oplus \mathrm{Cok}((1_k - X_k)(1_k + X_k^{(n+1)/2})) & (n \notin 2\boldsymbol{Z}) \\ (\mathrm{Cok}(1_k - X_k^{n+1}))^{(n-2)/2} \oplus \mathrm{Cok}(1_k - X_k) & (n \in 2\boldsymbol{Z}) \end{cases}.$$

Now we only have to calculate the direct summands appeared in the previous lemma. The results are the following, and using these, the part for $\boldsymbol{Z}A_n/\langle\tau^k\rangle$ of the main theorem is proved.

Lemma 16. [1, Lemma 2.8, Lemma 2.12] *We have* $\mathrm{Cok}(1_k - X_k) \cong \boldsymbol{Z}$, $\mathrm{Cok}(1_k - X_k^{n+1}) \cong \boldsymbol{Z}^d$ *and if* $n \notin 2\boldsymbol{Z}$,

$$\mathrm{Cok}((1_k - X_k)(1_k + X_k^{(n+1)/2})) \cong \begin{cases} \boldsymbol{Z} \oplus (\boldsymbol{Z}/2\boldsymbol{Z})^{d-1} & (r \in 2\boldsymbol{Z}) \\ \boldsymbol{Z}^{(d+2)/2} & (r \notin 2\boldsymbol{Z}) \end{cases}.$$

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