COHEN-MONTGOMERY DUALITY FOR BIMODULES AND ITS APPLICATIONS TO EQUIVALENCES GIVEN BY BIMODULES

HIDETO ASASHIBA

Abstract. Let G be a group and $\Bbbk$ be a commutative ring. We define a G-invariant bimodule ${}_SM_R$ over G-categories R, S and a G-graded bimodule ${}_BN_A$ over G-graded categories A, B, and introduce the orbit bimodule M/G and the smash product bimodule $N\#G$. We will show that these constructions are inverses to each other. This will be apply to Morita equivalences and stable equivalences of Morita type.

Introduction

We fix a commutative ring $\Bbbk$ and a group G. To include infinite coverings of $\Bbbk$-algebras into consideration we usually regard $\Bbbk$-algebras as locally bounded $\Bbbk$-categories with finite objects, and we will work with small $\Bbbk$-categories. For small $\Bbbk$-categories R and S with G-actions we introduce *G-invariant* S-R-bimodules and their category denoted by $S\text{-}\mathrm{Mod}^G\text{-}R$, and denote by R/G the orbit category of R by G, which is a small G-graded $\Bbbk$-category. For small G-graded $\Bbbk$-categories A and B we introduce *G-graded* B-A-bimodules and their category denoted by $B\text{-}\mathrm{Mod}_G\text{-}A$, and denote by $A\#G$ the smash product of A and G, which is a small $\Bbbk$-category with G-action. Then the Cohen-Montgomery duality theorem [4, 2] says that we have equivalences $(R/G)\#G \simeq R$ and $(A\#G)/G \simeq A$, by which we identify these pairs (see also [3]). Here we introduce functors $(\text{-})/G : S\text{-}\mathrm{Mod}^G\text{-}R \to (S/G)\text{-}\mathrm{Mod}_G\text{-}(R/G)$ and $(\text{-})\#G : A\text{-}\mathrm{Mod}_G\text{-}B \to (A\#G)\text{-}\mathrm{Mod}^G\text{-}(B\#G)$, and show that they are equivalences and quasi-inverses to each other (by applying $A := R/G$, $R := A\#G$, etc.), have good properties with tensor products and preserve projectivity of bimodules. We apply this to equivalences given by bimodules such as Morita equivalences, stable equivalences of Morita type to have theorems such as for stable equivalences of Morita type:

Theorem. (1) *There exists a "G-invariant stable equivalence of Morita type" between R and S if and only if there exists a "G-graded stable equivalence of Morita type" between R/G and S/G.*

(2) *There exists a "G-graded stable equivalence of Morita type" between A and B if and only if there exists a "G-invariant stable equivalence of Morita type" between $A\#G$ and $B\#G$.*

We note that a G-invariant (resp. G-graded) stable equivalence of Morita type is defined to be a usual stable equivalence of Morita type with additional properties, and does not mean an equivalence between stable categories of G-invariant (resp. G-graded) modules (see section 6 for detail).

The detailed version of this paper will be submitted for publication elsewhere.

For a category R we denote the class of objects and morphisms of R by R_0 and R_1, respectively. We sometimes write $x \in R$ for $x \in R_0$. We first recall definitions of G-categories and their 2-category G-**Cat**.

Definition 1.1. (1) A $\Bbbk$-category with a *G-action*, or a *G-category* for short, is a pair (R, X) of a category R and a group homomorphism $X\colon G \to \operatorname{Aut}(R)$. We often write ax for $X(a)x$ for all $a \in G$ and $x \in R_0 \cup R_1$ if there seems to be no confusion.

(2) Let $R = (R, X)$ and $R' = (R', X')$ be G-categories. Then a *G-equivariant* functor from R to R' is a pair (E, ρ) of a $\Bbbk$-functor $E\colon R \to R'$ and a family $\rho = (\rho_a)_{a\in G}$ of natural isomorphisms $\rho_a\colon X'_aE \Rightarrow EX_a$ $(a \in G)$ such that the diagrams

$$\begin{tikzcd} X'_{ba}E = X'_bX'_aE \arrow[r, Rightarrow, "X'_b\rho_a"] \arrow[dr, Rightarrow, "\rho_{ba}"'] & X'_bEX_a \arrow[d, Rightarrow, "\rho_b X_a"] \\ & EX_{ba} = EX_bX_a \end{tikzcd}$$

commute for all $a, b \in G$.

(3) A $\Bbbk$-functor $E\colon R \to R'$ is called a *strictly G-equivariant* functor if $(E, (\mathbb{1}_E)_{a\in G})$ is a G-equivariant functor, i.e., if $X'_aE = EX_a$ for all $a \in G$.

(4) Let $(E, \rho), (E', \rho')\colon R \to R'$ be *G-equivariant* functors. Then a *morphism* from (E, ρ) to (E', ρ') is a natural transformation $\eta\colon E \Rightarrow E'$ such that the diagrams

$$\begin{tikzcd} X'_aE \arrow[r, Rightarrow, "\rho_a"] \arrow[d, Rightarrow, "X'_a\eta"'] & EX_a \arrow[d, Rightarrow, "\eta X_a"] \\ X'_aE' \arrow[r, Rightarrow, "\rho'_a"'] & E'X_a \end{tikzcd}$$

commute for all $a \in G$.

These data define a 2-category G-**Cat** of small G-categories.

Next we recall definitions of G-graded categories and their 2-category G-**GrCat**.

Definition 1.2. (1) A *G-graded* $\Bbbk$-category is a category A together with a family of direct sum decompositions $A(x, y) = \bigoplus_{a\in G} A^a(x, y)$ $(x, y \in A)$ of $\Bbbk$-modules such that $A^b(y, z) \cdot A^a(x, y) \subseteq A^{ba}(x, z)$ for all $x, y \in A$ and $a, b \in G$. It is easy to see that $\mathbb{1}_x \in A^1(x, x)$ for all $x \in A_0$.

(2) A *degree-preserving* functor is a pair (H, r) of a $\Bbbk$-functor $H\colon A \to B$ of G-graded categories and a map $r\colon A_0 \to G$ such that

$$H(A^{r_y a}(x, y)) \subseteq B^{ar_x}(Hx, Hy)$$

(or equivalently $H(A^a(x, y)) \subseteq B^{r_y^{-1}ar_x}(Hx, Hy)$) for all $x, y \in A$ and $a \in G$. This r is called a *degree adjuster* of H.

(3) A $\Bbbk$-functor $H\colon A \to B$ of G-graded categories is called a *strictly* degree-preserving functor if $(H, 1)$ is a degree-preserving functor, where 1 denotes the constant map $A_0 \to G$ with value $1 \in G$, i.e., if $H(A^a(x, y)) \subseteq B^a(Hx, Hy)$ for all $x, y \in A$ and $a \in G$.

(4) Let $(H, r), (I, s)\colon A \to B$ be degree-preserving functors. Then a natural transformation $\theta\colon H \Rightarrow I$ is called a *morphism* of degree-preserving functors if $\theta x \in B^{s_x^{-1} r_x}(Hx, Ix)$ for all $x \in A$.

These data define a 2-category G-**GrCat** of small G-graded categories.

Finally we recall definitions of orbit categories and smash products, and their relationships.

Definition 1.3. Let R be a G-category. Then the *orbit category* R/G of R by G is a category defined as follows.

- $(R/G)_0 := R_0$;
- For any $x, y \in G$, $(R/G)(x, y) := \bigoplus_{a \in G} R(ax, y)$; and
- For any $x \xrightarrow{f} y \xrightarrow{g} z$ in R/G, $gf := (\sum_{a,b \in G; ba=c} g_b \cdot b(f_a))_{c \in G}$.
- For each $x \in (R/G)_0$ its identity $\mathbb{1}_x := \mathbb{1}_x^{R/G}$ in R/G is given by $\mathbb{1}_x = (\delta_{a,1}\mathbb{1}_x^R)_{a \in G}$, where $\mathbb{1}_x^R$ is the identity of x in R.

By setting $(R/G)^a(x, y) := R(ax, y)$ for all $x, y \in R_0$ and $a \in G$, the decompositions $(R/G)(x, y) = \bigoplus_{a \in G}(R/G)^a(x, y)$ makes R/G a G-graded category.

Definition 1.4. Let A be a G-graded category. Then the *smash product* $A\#G$ is a category defined as follows.

- $(A\#G)_0 := A_0 \times G$, we set $x^{(a)} := (x, a)$ for all $x \in A$ and $a \in G$.
- $(A\#G)(x^{(a)}, y^{(b)}) := A^{b^{-1}a}(x, y)$ for all $x^{(a)}, y^{(b)} \in A\#G$.
- For any $x^{(a)}, y^{(b)}, z^{(c)} \in A\#G$ the composition is given by the following commutative diagram

$$\begin{array}{ccc} (A\#G)(y^{(b)}, z^{(c)}) \times (A\#G)(x^{(a)}, y^{(b)}) & \longrightarrow & (A\#G)(x^{(a)}, z^{(c)}) \\ \| & & \| \\ A^{c^{-1}b}(y, z) \times A^{b^{-1}a}(x, y) & \longrightarrow & A^{c^{-1}a}(x, z), \end{array}$$

  where the lower horizontal homomorphism is given by the composition of A.
- For each $x^{(a)} \in (A\#G)_0$ its identity $\mathbb{1}_{x^{(a)}}$ in $A\#G$ is given by $\mathbb{1}_x \in A^1(x, x)$.

$A\#G$ has a free G-action defined as follows: For each $c \in G$ and $x^{(a)} \in A\#G$, $cx^{(a)} := x^{(ca)}$; and for each $f \in (A\#G)(x^{(a)}, y^{(b)}) = A^{b^{-1}a}(x, y) = (A\#G)(x^{(ca)}, y^{(cb)})$, $cf := f$.

The following two propositions were proved in [1].

Proposition 1.5 ([1, Proposition 5.6])**.** *Let A be a G-graded category. Then there is a strictly degree-preserving equivalence $\omega_A\colon A \to (A\#G)/G$ of G-graded categories.*

Proposition 1.6 ([1, Theorem 5.10])**.** *Let R be a category with a G-action. Then there is a G-equivariant equivalence $\varepsilon_R\colon R \to (R/G)\#G$.*

In fact, the orbit category construction and the smash product construction can be extended to 2-functors (-)$/G\colon G$-**Cat** $\to G$-**GrCat** and (-)$\#G\colon G$-**GrCat** $\to G$-**Cat**, respectively, and they are inverses to each other as stated in the following theorem, where $\omega := (\omega_A)_A$ and $\varepsilon := (\varepsilon_R)_R$ are 2-natural isomorhisms.

Theorem 1.7 ([2, Theorem 7.5]). *(-)/G is strictly left 2-adjoint to (-)#G and they are mutual 2-quasi-inverses.*

Remark 1.8. $\omega_A \colon A \to (A\#G)/G$ above is an equivalence in the 2-category G-**GrCat** and $\varepsilon_R \colon R \to (R/G)\#G$ above is an equivalence in the 2-category G-**Cat**. By these equivalences we identify $(A\#G)/G$ with A, and $(R/G)\#G$ with R in the following sections.

2. G-invariant bimodules and G-graded bimodues

Definition 2.1. Let $R = (R, X)$ and $S = (S, Y)$ be small $\Bbbk$-categories with G-actions.

(1) A *G-invariant* S-R-bimodule is a pair (M, ϕ) of an S-R-bimodule M and a family $\phi := (\phi_a)_{a\in G}$ of natural transformations $\phi_a \colon M \to M(X(a)(\text{-}), Y(a)(\text{-}))$, where $\phi_a = (\phi_a(x, y))_{(x,y)\in R_0\times S_0}$, $\phi_a(x, y) \colon M(x, y) \to M(ax, ay)$ is in Mod $\Bbbk$, such that the following diagram commutes for all $a, b \in G$ and all $(x, y) \in R_0 \times S_0$:

$$\begin{tikzcd} M(x,y) \arrow[r, "\phi_a(x,y)"] \arrow[dr, "\phi_{ba}(x,y)"'] & M(ax,ay) \arrow[d, "\phi_b(ax,ay)"] \\ & M(bax,bay). \end{tikzcd}$$

(2) Let (M, ϕ) and (N, ψ) be G-invariant S-R-bimodules. A *morphism* $(M, \phi) \to (N, \psi)$ is an S-R-bimodule morphism $F \colon M \to N$ such that the following diagram commutes for all $a \in G$ and all $(x, y) \in R_0 \times S_0$:

$$\begin{tikzcd} M(x,y) \arrow[r, "\phi_a(x,y)"] \arrow[d, "F(x,y)"'] & M(ax,ay) \arrow[d, "F(ax,ay)"] \\ N(x,y) \arrow[r, "\psi_a(x,y)"'] & N(ax,ay). \end{tikzcd}$$

(3) The class of all G-invariant S-R-bimodules together with all morphisms between them forms a $\Bbbk$-category denoted by S-Mod^G-R.

Remark 2.2. The commutativity of the diagram in (1) above for $a = b = 1$ shows that $\phi_1 = \mathbb{1}_M$, which also shows that $\phi_a(x, y)^{-1} = \phi_{a^{-1}}(ax, ay)$ for all $a \in G$ and all $(x, y) \in R_0 \times S_0$.

Definition 2.3. Let A and B be G-graded small $\Bbbk$-categories.

(1) A G-graded B-A-bimodule is a B-A-bimodule M together with decompositions $M(x, y) = \bigoplus_{a\in G} M^a(x, y)$ in Mod $\Bbbk$ for all $(x, y) \in A_0 \times B_0$ such that

$$B^c(y, y') \cdot M^a(x, y) \cdot A^b(x', x) \subseteq M^{cab}(x', y')$$

for all $a, b, c \in G$ and all $x, x' \in A_0, y, y' \in B_0$.

(2) Let M and N be G-graded B-A-bimodules. Then a *morphism* $M \to N$ is a B-A-bimodule morphism $F \colon M \to N$ such that $F(M^a(x, y)) \subseteq N^a(Fx, Fy)$ for all $a \in G$ and all $(x, y) \in A_0 \times B_0$.

(3) The class of all G-graded B-A-bimodules together with all morphisms between them forms a $\Bbbk$-category denoted by B-Mod_G-A.

Throughout this section $R = (R, X)$ and $S = (S, Y)$ are small $\Bbbk$-categories with G-actions, and $E\colon R \to R/G$ and $F\colon S \to S/G$ the canonical G-covering, respectively.

Definition 3.1. (1) Let $M = (M, \phi)$ be a G-invariant S-R-bimodule. Then we form a G-graded S/G-R/G-bimodule M/G as follows which we call the *orbit bimodule* of M by G:

- For each $(x, y) \in (R/G)_0 \times (S/G)_0 = R_0 \times S_0$ we set
$$(M/G)(x, y) := \bigoplus_{a\in G} M(ax, y). \tag{3.1}$$
- For each $(x, y), (x', y') \in (R/G)_0 \times (S/G)_0 = R_0 \times S_0$ and each $(r, s) \in (R/G)(x', x) \times (S/G)(y, y')$ we define a morphism
$$(M/G)(r, s)\colon (M/G)(x, y) \to (M/G)(x', y')$$
in $\operatorname{Mod}\Bbbk$ by
$$(M/G)(r, s)(m) := s \cdot m \cdot r := \left(\sum_{cba=d} s_c \cdot \phi_c(m_b) \cdot cbr_a\right)_{d\in G} \tag{3.2}$$
for all $r = (r_a)_{a\in G} \in \bigoplus_{a\in G} R(ax', x), m = (m_b)_{b\in G} \in \bigoplus_{b\in G} M(ax, y)$, and $s = (s_c)_{c\in G} \in \bigoplus_{c\in G} S(cy, y')$. By the naturality of ϕ_a $(a \in G)$ we easily see that (3.2) defines an (S/G)-(R/G)-bimodule structure on M/G.
- We set $M^a(x, y) := M(ax, y)$ for all $a \in G$ and all $(x, y) \in R_0 \times S_0$. We easily see that this defines a G-grading on M/G by (3.1) and (3.2).

(2) Let $f\colon M \to N$ be in S-Mod^G-R. For each $(x, y) \in R_0 \times S_0$ we set
$$(f/G)(x, y) := \bigoplus_{a\in G} f(ax, y).$$
Then as is easily seen $f/G := (f/G(x, y))_{(x,y)\in R_0\times S_0}$ turns out to be a morphism $M/G \to N/G$ in (S/G)-Mod_G-(R/G).

(3) It is easy to see that (1) together with (2) above defines a $\Bbbk$-functor
$$(\text{-})/G\colon S\text{-}\operatorname{Mod}^G\text{-}R \to (S/G)\text{-}\operatorname{Mod}_G\text{-}(R/G).$$

Lemma 3.2. *By regarding R/G as a left R-module and a right R-module via the canonical G-covering functor $E\colon R \to R/G$, we have*
$$R/G \otimes_R R/G \cong R/G \otimes_{R/G} R/G \cong R/G$$
as (R/G)-(R/G)-bimodules.

Proposition 3.3. *Let M be a G-invariant S-R-bimodule. Then*

(1) $M \otimes_R (R/G) \cong {}_FM/G$ *as S-(R/G)-bimodules; and*
(2) $(S/G) \otimes_S M \cong M/G_E$ *as (S/G)-R-bimodules.*

Hence in particular we have isomorphisms of S-R-bimodules

(3) $M \otimes_R (R/G)_E \cong {}_FM/G_E \cong {}_F(S/G) \otimes_S M$

and an isomorphism of G-graded (S/G)-(R/G)-bimodules

(4) $(S/G) \otimes_S M \otimes_R (R/G) \cong M/G$.

Proposition 3.4. *Let $T = (T, Z)$ be a small $\Bbbk$-category with G-action, and ${}_SM_R$, ${}_TN_S$ be G-invariant bimodules. Then*

(1) *${}_T(N \otimes_S M)_R$ is a G-invariant bimodule.*
(2) *$(N \otimes_S M)/G \cong (N/G) \otimes_{S/G} (M/G)$ in (T/G)-Mod_G-(R/G).*

Proposition 3.5. *Let ${}_SP_R$ be a projective bimodule that is G-invariant. Then ${}_{(S/G)}P/G_{(R/G)}$ is a projective bimodule that is G-graded.*

Remark 3.6. In the proof above, note that in general we have

$$(S(w, \text{-}) \otimes_\Bbbk R(\text{-}, z))/G \not\cong (S/G) \otimes_S S(w, \text{-}) \otimes_\Bbbk R(\text{-}, z) \otimes_R (R/G)$$

because $S(w, \text{-}) \otimes_\Bbbk R(\text{-}, z)$ is not always G-invariant.

4. Smash product

Throughout this section A and B are G-graded small $\Bbbk$-categories.

Definition 4.1. (1) Let M be a G-graded B-A-bimodule. Then we define a G-invariant $(B\#G)$-$(A\#G)$-bimodule $M\#G$ as follows, which we call the *smash product* of M and G:

- For each $(x^{(a)}, y^{(b)}) \in (A\#G)_0 \times (B\#G)_0$ we set
$$(M\#G)(x^{(a)}, y^{(b)}) := M^{b^{-1}a}(x, y).$$
- For each $(x^{(a)}, y^{(b)}), (x'^{(a')}, y'^{(b')}) \in (A\#G)_0 \times (B\#G)_0$ and each $(\alpha, \beta) \in (A\#G)(x'^{(a')}, x^{(a)}) \times (B\#G)(y^{(b)}, y'^{(b')}) = A^{a^{-1}a'}(x', x) \times B^{b'^{-1}b}(y, y')$ we define a morphism $(M\#G)(\alpha, \beta)$ in $\mathrm{Mod}\,\Bbbk$ by the following commutative diagram:
$$\begin{tikzcd} (M\#G)(x^{(a)}, y^{(b)}) \arrow[r, "(M\#G)(\alpha,\beta)"] \arrow[d, equal] & (M\#G)(x'^{(a')}, y'^{(b')}) \arrow[d, equal] \\ M^{b^{-1}a}(x, y) \arrow[r, "M(\alpha,\beta)"'] & M^{b'^{-1}a'}(x', y'). \end{tikzcd}$$
Since $M(\alpha, \beta)(m) \in M^{(b'^{-1}b)(b^{-1}a)(a^{-1}a')}(x', y') = M^{b'^{-1}a'}(x', y')$ for all $m \in M^{b^{-1}a}(x, y)$, the bottom morphism is well-defined. It is easy to verify that this makes $M\#G$ a $(B\#G)$-$(A\#G)$-bimodule.
- For each $(x^{(a)}, y^{(b)}) \in (A\#G)_0 \times (B\#G)_0$ and each $c \in G$ we define $\phi_c(x^{(a)}, y^{(b)})$ by the following commutative diagram:
$$\begin{tikzcd} (M\#G)(x^{(a)}, y^{(b)}) \arrow[r, "\phi_c(x^{(a)},y^{(b)})"] \arrow[d, equal] & (M\#G)(c \cdot x^{(a)}, c \cdot y^{(b)}) \arrow[d, equal] \\ M^{b^{-1}a}(x, y) \arrow[r, equal] & (M\#G)(x^{(ca)}, y^{(cb)}). \end{tikzcd}$$
Then by letting $\phi_c := (\phi_c(x^{(a)}, y^{(b)}))_{(x^{(a)}, y^{(b)})}$, and $\phi := (\phi_c)_{c \in G}$, we have a G-invariant $(B\#G)$-$(A\#G)$-bimodule $(M\#G, \phi)$.

(2) Let $f\colon M \to N$ be in B-Mod_G-A. For each $(x^{(a)}, y^{(b)}) \in (A\#G)_0 \times (B\#G)_0$ we define $(f\#G)(x^{(a)}, y^{(b)})$ by the commutative diagram

$$\begin{tikzcd}
(M\#G)(x^{(a)}, y^{(b)}) \arrow[r, "(f\#G)(x^{(a)}, y^{(b)})"] \arrow[d, equal] & (N\#G)(x^{(a)}, y^{(b)}) \arrow[d, equal] \\
M^{b^{-1}a}(x,y) \arrow[r, "f|_{M^{b^{-1}a}(x,y)}"'] & N^{b^{-1}a}(x,y).
\end{tikzcd}$$

Then as is easily seen $f\#G := ((f\#G)(x^{(a)}, y^{(b)}))_{(x^{(a)}, y^{(b)})}$ is a morphism $M\#G \to N\#G$ in the category $(B\#G)$-Mod^G-$(A\#G)$.

(3) It is easy to see that (1) together with (2) above defines a $\Bbbk$-functor

$$(\text{-})\#G\colon B\text{-}\mathrm{Mod}_G\text{-}A \to (B\#G)\text{-}\mathrm{Mod}^G\text{-}(A\#G).$$

Proposition 4.2. *Let C be a G-graded small $\Bbbk$-category, and ${}_BM_A, {}_CN_B$ G-graded bimodules. Then*

(1) *$N \otimes_B M$ is a G-graded C-A-bimodule.*

(2) *$(N \otimes_B M)\#G \cong (N\#G) \otimes_{B\#G} (M\#G)$ in $(C\#G)$-Mod^G-$(A\#G)$.*

Proposition 4.3. *Let ${}_BP_A$ be a projective bimodule that is G-graded. Then ${}_{B\#G}(P\#G)_{A\#G}$ is a projective bimodule that is G-invariant.*

5. Cohen-Montgomery duality for bimodules

Theorem 5.1. *Let R, S be small $\Bbbk$-categories with G-actions, and A, B be G-graded small $\Bbbk$-categories.*

(1) *The functor $(\text{-})/G\colon S$-Mod^G-$R \to (S/G)$-Mod_G-(R/G) is an equivalence, a quasi-inverse of which is given by the composite*

$$(S/G)\text{-}\mathrm{Mod}_G\text{-}(R/G) \xrightarrow{(\text{-})\#G} ((S/G)\#G)\text{-}\mathrm{Mod}^G\text{-}((R/G)\#G) \xrightarrow{\sim} S\text{-}\mathrm{Mod}^G\text{-}R.$$

(2) *The functor $(\text{-})\#G\colon B$-Mod_G-$A \to (B\#G)$-Mod^G-$(A\#G)$ is an equivalence, a quasi-invers of which is given by the composite*

$$(B\#G)\text{-}\mathrm{Mod}^G\text{-}(A\#G) \xrightarrow{(\text{-})/G} ((B\#G)/G)\text{-}\mathrm{Mod}_G\text{-}((A\#G)/G) \xrightarrow{\sim} B\text{-}\mathrm{Mod}_G\text{-}A.$$

(3) *In particular, for each G-invariant bimodule ${}_RM_S$ we have $(M/G)\#G \cong M$ as S-R-bimodules, and for each G-graded bimodule ${}_BM_A$ we have $(M\#G)/G \cong M$ as B-A-bimodules.*

6. Applications

Definition 6.1. Let R, S be small $\Bbbk$-categories with G-actions, and A, B be G-graded small $\Bbbk$-categories.

(1) A pair $({}_SM_R, {}_RN_S)$ of bimodules is said to give a *G-invariant stable equivalence of Morita type* between R and S if ${}_SM, M_R, {}_RN, N_S$ are projective modules and ${}_SM_R, {}_RN_S$ are G-invariant bimodules such that $N \otimes_S M \cong R \oplus {}_RP_R$ and

$M \otimes_R N \cong S \oplus {}_SQ_S$ as G-invariant bimodules for some projective bimodules ${}_RP_R, {}_SQ_S$ that are G-invariant.

(1′) A pair $({}_SM_R, {}_RN_S)$ of bimodules is said to give a *G-invariant Morita equivalence* between R and S if it gives a G-invariant stable equivalence of Morita type with $P = 0 = Q$ in (1) above.

(2) A pair $({}_BM_A, {}_AN_B)$ of bimodules is said to give a *G-graded stable equivalence of Morita type* between A and B if ${}_BM, M_A, {}_AN, N_B$ are projective modules and ${}_BM_A, {}_AN_B$ are G-graded bimodules such that $N \otimes_B M \cong A \oplus {}_AP_A$ and $M \otimes_A N \cong B \oplus {}_BQ_B$ as G-graded bimodules for some projective bimodules ${}_AP_A, {}_BQ_B$ that are G-graded.

(2′) A pair $({}_BM_A, {}_AN_B)$ of bimodules is said to give a *G-graded Morita equivalence* between A and B if it gives a G-graded stable equivalence of Morita type with $P = 0 = Q$ in (2) above.

Theorem 6.2. *Let R, S be small $\Bbbk$-categories with G-actions, and A, B be G-graded small $\Bbbk$-categories.*

(1) *A pair $({}_SM_R, {}_RN_S)$ of bimodules gives a G-invariant stable equivalence of Morita type between R and S if and only if the pair $(M/G, N/G)$ gives a G-graded stable equivalence of Morita type between R/G and S/G.*

(1′) *A pair $({}_SM_R, {}_RN_S)$ of bimodules gives a G-invariant Morita equivalence between R and S if and only if the pair $(M/G, N/G)$ gives a G-graded Morita equivalence between R/G and S/G.*

(2) *A pair $({}_BM_A, {}_AN_B)$ of bimodules gives a G-graded stable equivalence of Morita type between A and B if and only if the pair $(M\#G, N\#G)$ gives a G-invariant stable equivalence of Morita type between $A\#G$ and $B\#G$.*

(2′) *A pair $({}_BM_A, {}_AN_B)$ of bimodules gives a G-graded Morita equivalence between A and B if and only if the pair $(M\#G, N\#G)$ gives a G-invariant Morita equivalence between $A\#G$ and $B\#G$.*

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Department of Mathematics,
Faculty of Science,
Shizuoka University,
836 Ohya, Suruga-ku,
Shizuoka, 422-8529, Japan

E-mail address: asashiba.hideto@shizuoka.ac.jp