STABLE DEGENERATIONS OF COHEN-MACAULAY MODULES OVER SIMPLE SINGULARITIES OF TYPE (A_n)

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Abstract. We study the stable degeneration problem for Cohen-Macaulay modules over simple singularities of type (A_n). We prove that the stable hom order is actually a partial order over the ring and are able to show that the stable degenerations can be controlled by the stable hom order.

1. Introduction

The concept of degenerations of modules introduced in representation theory for studying the structure of the module variety over a finite dimensional algebra. Classically Bongartz [1] investigated the degeneration problem of modules over an artinian algebra in relation with the Auslander-Reiten quiver. In [11], Zwara gave a complete description of degenerations of modules over representation finite algebras by using some order relations for modules known as the hom order, the degeneration order and the extension order. Now a theory of degenerations is considered for not only module categories, but derived categories [5] or stable categories [10], more generally, triangulated categories [7].

Let R be a commutative Gorenstein local k-algebra which is not necessary finite dimensional. Yoshino [10] introduced a notion of the stable analogue of degenerations of (maximal) Cohen-Macaulay R-module in the stable category $\underline{\mathrm{CM}}(R)$. The author [4] give a complete description of degenerations of Cohen-Macaulay modules over a ring of even dimensional simple singularity of type (A_n) by using the description of stable degenerations over it. Hence it is also important for the study of degeneration problem to investigate the description of stable degenerations.

The purpose of this paper is to describe stable degenerations of Cohen-Macaulay modules over simple singularities of type (A_n).

$$k[[x_0, x_1, x_2, \cdots, x_d]]/(x_0^{n+1} + x_1^2 + x_2^2 + \cdots + x_d^2).$$

First we consider an order relation on $\underline{\mathrm{CM}}(R)$ which is the stable analogue of the hom order and show that the order $\leq_{\underline{hom}}$ is a partial order on $\underline{\mathrm{CM}}(R)$ if n is of odd dimensional. By using the stable analogue of the argument over finite dimensional algebras in [11], we can describe stable degenerations of Cohen-Macaulay modules over the ring in terms of the stable hom order.

2. Stable hom order

Throughout the paper R is a commutative Henselian Gorenstein local ring that is k-algebra where k is an algebraically closed field of characteristic 0. For a finitely generated

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R-module M, we say that M is a Cohen-Macaulay R-module if

$$\mathrm{Ext}^i_R(M, R) = 0 \quad \text{for any } i > 0.$$

We denote by CM(R) the category of Cohen-Macaulay R-modules with all R-homomorphisms. And we also denote by $\underline{\mathrm{CM}}(R)$ the stable category of CM(R). The objects of $\underline{\mathrm{CM}}(R)$ are the same as those of CM(R), and the morphisms of $\underline{\mathrm{CM}}(R)$ are elements of $\underline{\mathrm{Hom}}_R(M, N) = \mathrm{Hom}_R(M, N)/P(M, N)$ for $M, N \in \underline{\mathrm{CM}}(R)$, where $P(M, N)$ denote the set of morphisms from M to N factoring through free R-modules. For a Cohen-Macaulay module M we denote it by $\underline{M}$ to indicate that it is an object of $\underline{\mathrm{CM}}(R)$. For a finitely generated R-module M, take a free resolution

$$\cdots \to F_1 \xrightarrow{d} F_0 \to M \to 0.$$

We denote Imd by ΩM. We note that this defines the functor giving an auto-equivalence of $\underline{\mathrm{CM}}(R)$. It is known that $\underline{\mathrm{CM}}(R)$ has a structure of a triangulated category with the shift functor defined by the functor Ω^{-1}. We recommend the reader to [2, Chapter 1], [8, Section 4] for the detail. Since R is Henselian, CM(R), hence $\underline{\mathrm{CM}}(R)$, is a Krull-Schmidt category, namely each object can be decomposed into indecomposable objects up to isomorphism uniquely.

Definition 1. We say that $(R, \mathfrak{m})$ is an isolated singularity if each localization $R_\mathfrak{p}$ is regular for each prime ideal $\mathfrak{p}$ with $\mathfrak{p} \neq \mathfrak{m}$.

We say that CM(R) (resp. $\underline{\mathrm{CM}}(R)$) admits AR sequences (resp. AR triangles) if there exists an AR sequence (reap. AR triangle) ending in X (resp. $\underline{X}$) for each indecomposable Cohen-Macaulay R-module X. If R is an isolated singularity, CM(R) admits AR sequences (see [8, Theorem 3.2]). Hence, $\underline{\mathrm{CM}}(R)$ also admits AR triangles. We also say that R is of finite representation type if there are only a finite number of isomorphism classes of indecomposable Cohen-Macaulay R-modules. We note that if R is of finite representation type, then R is an isolated singularity (cf. [8, Chapter 3.]).

Lemma 2. [8, Lemma 3.9] *Let M and N be finitely generated R-modules. Then we have a functorial isomorphism*

$$\underline{\mathrm{Hom}}_R(M, N) \cong \mathrm{Tor}^R_1(\mathrm{Tr}M, N).$$

Here TrM *is an Auslander transpose of M.*

According to Lemma 2, $\underline{\mathrm{Hom}}_R(M, N)$ is of finite dimensional as a k-module for M, $N \in \mathrm{CM}(R)$ if R is an isolated singularity. Thus the following definition makes sense.

Definition 3. For M, $N \in \mathrm{CM}(R)$ we define $\underline{M} \leq_{hom} \underline{N}$ if $[\underline{X}, \underline{M}] \leq [\underline{X}, \underline{N}]$ for each $\underline{X} \in \underline{\mathrm{CM}}(R)$. Here $[\underline{X}, \underline{M}]$ is an abbreviation of $\dim_k \underline{\mathrm{Hom}}_R(X, M)$.

Now let us consider the full subcategory of the functor category of $\underline{\mathrm{CM}}(R)$ which is called the Auslander category. The Auslander category mod($\underline{\mathrm{CM}}(R)$) is the category whose objects are finitely presented contravariant additive functors from $\underline{\mathrm{CM}}(R)$ to the category of Abelian groups and whose morphisms are natural transformations between functors.

Lemma 4. [8, Theorem 13.7] *A group homomorphism*

$$\gamma : \mathrm{G}(\mathrm{CM}(R)) \to K_0(\mathrm{mod}(\mathrm{CM}(R))),$$

defined by $\gamma(M) = [\mathrm{Hom}_R(\ ,M)]$ *for* $M \in \mathrm{CM}(R)$*, is injective. Here* $\mathrm{G}(\mathrm{CM}(R))$ *is a free Abelian group* $\bigoplus \mathbb{Z}\cdot X$*, where* X *runs through all isomorphism classes of indecomposable objects in* $\mathrm{CM}(R)$*.*

We denote by $\underline{\mathrm{mod}}(\mathrm{CM}(R))$ the full subcategory $\mathrm{mod}(\mathrm{CM}(R))$ consisting of functors F with $F(R) = 0$. Note that every object $F \in \underline{\mathrm{mod}}(\mathrm{CM}(R))$ is obtained from a short exact sequence in $\mathrm{CM}(R)$. Namely we have the short exact sequence $0 \to L \to M \to N \to 0$ such that

$$0 \to \mathrm{Hom}_R(\ ,L) \to \mathrm{Hom}_R(\ ,M) \to \mathrm{Hom}_R(\ ,N) \to F \to 0$$

is exact in $\mathrm{mod}(\mathrm{CM}(R))$. If $0 \to Z \to Y \to X \to 0$ is an AR sequence in $\mathrm{CM}(R)$, then the functor S_X defined by an exact sequence

$$0 \to \mathrm{Hom}_R(\ ,Z) \to \mathrm{Hom}_R(\ ,Y) \to \mathrm{Hom}_R(\ ,X) \to S_X \to 0$$

is a simple object in $\mathrm{mod}(\mathrm{CM}(R))$ and all the simple objects in $\mathrm{mod}(\mathrm{CM}(R))$ are obtained in this way from AR sequences.

Proposition 5. [3, Lemma 2.8] *If* R *is of finite representation type, then we have the equality in* $K_0(\underline{\mathrm{mod}}(\mathrm{CM}(R)))$

$$[\underline{\mathrm{Hom}}_R(-,M)] = \sum_{X_i \in \mathrm{ind}\mathrm{CM}(R)} [\underline{X_i}, \underline{M}] \cdot [S_{X_i}]$$

for each $M \in \mathrm{CM}(R)$*.*

Proof. Since $\underline{\mathrm{Hom}}_R(-,M)$ is an object in $\underline{\mathrm{mod}}(\mathrm{CM}(R))$ we have a filtration by subjects $0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = F$ such that each F_i/F_{i-1} is a simple object S_X in $\mathrm{mod}(\mathrm{CM}(R))$ (cf. [8, (13.7.4)]). Hence we have the equality in $K_0(\underline{\mathrm{mod}}(\mathrm{CM}(R)))$:

$$[\underline{\mathrm{Hom}}_R(-,M)] = \sum_{X_i \in \mathrm{ind}\mathrm{CM}(R)} c_i \cdot [S_{X_i}].$$

By using a property of AR sequences, one can show that $c_i = [\underline{X_i}, \underline{M}]$, so that we obtain the equation. ☐

Combining Proposition 5 with Lemma 4, we have

Theorem 6. [3, Theorem 2.9] *Let* R *be of finite representation type and* M *and* N *be Cohen-Macaulay* R*-modules. Suppose that* $[\underline{X}, \underline{M}] = [\underline{X}, \underline{N}]$ *for each* $\underline{X} \in \underline{\mathrm{CM}}(R)$*. Then* $\underline{M} \oplus \underline{\Omega M} \cong \underline{N} \oplus \underline{\Omega N}$*.*

It immediately follows from the theorem that

Corollary 7. *Let* R *be of finite representation type. Suppose that* $\underline{U} \cong \underline{U}[-1]$ *for each indecomposable Cohen-Macaulay* R*-module* U*. Then* $[\underline{X}, \underline{M}] = [\underline{X}, \underline{N}]$ *for each* $\underline{X} \in \underline{\mathrm{CM}}(R)$ *if and only if* $\underline{M} \cong \underline{N}$*. Particularly,* $\leq_{hom}$ *is a partial order on* $\underline{\mathrm{CM}}(R)$*.*

Example 8. Let R be a one dimensional simple singularity of type (A_n), that is $R = k[[x,y]]/(x^{n+1}+y^2)$. If n is an even integer, one can show that X is isomorphic to ΩX up to free summed for each $X \in \mathrm{CM}(R)$. See [8, Proposition 5.11]. Thus $\leq_{\underline{hom}}$ is a partial order on $\underline{\mathrm{CM}}(R)$ if n is an even integer.

If n is an odd integer, we have indecomposable modules $X \in \mathrm{CM}(R)$ such that $\underline{X} \not\cong \underline{X}[-1]$. In fact, let $N_\pm = R/(x^{(n+1)/2} \pm \sqrt{-1}y)$. Then N_+ (resp. N_-) is a Cohen-Macaulay R-module which is isomorphic to ΩN_- (resp. ΩN_+), so that $\underline{N_+} \not\cong \underline{N_+}[-1]$ (resp. $\underline{N_-} \not\cong \underline{N_-}[-1]$). Although we can also show that $\leq_{\underline{hom}}$ is a partial order on $\underline{\mathrm{CM}}(R)$ even if n is an odd integer.

Proposition 9. [3, Proposition 2.12] *Let* $R = k[[x,y]]/(x^{n+1}+y^2)$. *Then* $[\underline{X}, \underline{M}] = [\underline{X}, \underline{N}]$ *for each* $\underline{X} \in \underline{\mathrm{CM}}(R)$ *if and only if* $\underline{M} \cong \underline{N}$.

Remark 10. The stable hom order $\leq_{\underline{hom}}$ is not always a partial order on $\underline{\mathrm{CM}}(R)$ even if the base ring R is a simple singularity of type (A_n). See [3, Remark 2.13].

3. Stable degeneration of Cohen-Macaulay modules

In this section, we shall describe the stable degenerations of Cohen-Macaulay modules over simple singularities over type (A_n), namely

$$k[[x_0, x_1, x_2, \cdots, x_d]]/(x_0^{n+1} + x_1^2 + x_2^2 + \cdots + x_d^2). \tag{3.1}$$

First let us recall the definition of stable degenerations of Cohen-Macaulay modules.

Definition 11. [10, Definition 4.1] Let $\underline{M}, \underline{N} \in \underline{\mathrm{CM}}(R)$. We say that $\underline{M}$ stably degenerates to $\underline{N}$ if there exists a Cohen-Macaulay module $Q \in \underline{\mathrm{CM}}(R \otimes_k V)$ such that $\underline{Q[1/t]} \cong \underline{M \otimes_k K}$ in $\underline{\mathrm{CM}}(R \otimes_k K)$ and $\underline{Q \otimes_V V/tV} \cong \underline{N}$ in $\underline{\mathrm{CM}}(R)$.

It is known that the ring (3.1) is of finite representation type, so that it is an isolated singularity. If a ring is an isolated singularity, there is a nice characterization of stable degenerations.

Theorem 12. [10, Theorem 5.1, 6.1] *Consider the following three conditions for Cohen-Macaulay R-modules M and N:*

(1) *$M \oplus P$ degenerates to $N \oplus Q$ for some free R-modules P, Q.*
(2) *$\underline{M}$ stably degenerates to $\underline{N}$.*
(3) *There is a triangle*

$$\underline{Z} \longrightarrow \underline{M} \oplus \underline{Z} \longrightarrow \underline{N} \longrightarrow \underline{Z}[1]$$

in $\underline{\mathrm{CM}}(R)$.

If R is an isolated singularity, then (2) *and* (3) *are equivalent. Moreover, if R is artinian, the conditions* (1), (2) *and* (3) *are equivalent.*

We state order relations with respect to stable degenerations and triangles.

Definition 13. [4, Definition 3.2., 3.3.] Let M and $N \in \mathrm{CM}(R)$.

(1) We denote by $\underline{M} \leq_{st} \underline{N}$ if $\underline{N}$ is obtained from $\underline{M}$ by iterative stable degenerations, i.e. there is a sequence of Cohen-Macaulay R-modules $\underline{L_0}, \underline{L_1}, \ldots, \underline{L_r}$ such that $\underline{M} \cong \underline{L_0}$, $\underline{N} \cong \underline{L_r}$ and each $\underline{L_i}$ stably degenerates to $\underline{L_{i+1}}$ for $0 \leq i < r$.

(2) We say that $\underline{M}$ stably degenerates by a triangle to $\underline{N}$, if there is a triangle of the form $\underline{U} \to \underline{M} \to \underline{V} \to \underline{U}[1]$ in $\underline{\mathrm{CM}}(R)$ such that $\underline{U} \oplus \underline{V} \cong \underline{N}$. We also denote by $\underline{M} \leq_{tri} \underline{N}$ if $\underline{N}$ is obtain from $\underline{M}$ by iterative stable degenerations by a triangle.

Remark 14. It has shown in [10] that the stable degeneration order is a partial order. Moreover if there is a triangle $\underline{U} \to \underline{M} \to \underline{V} \to \underline{U}[1]$, then we can show that $\underline{M}$ stably degenerates to $\underline{U} \oplus \underline{V}$ (cf. [4, Remark 3.4. (2)]). Hence $\underline{M} \leq_{tri} \underline{N}$ induces $\underline{M} \leq_{st} \underline{N}$. It also follows from Theorem 12 that $\underline{M} \leq_{st} \underline{N}$ induces that $\underline{M} \leq_{\underline{hom}} \underline{N}$.

Let R be an one dimensional simple singularity of type (A_n). That is $R = k[[x,y]]/(x^{n+1} + y^2)$. As stated in [8, Proposition 5.11], if n is an even integer, the set of ideals of R

$$\{ \ I_i = (x^i, y) \mid 1 \leq i \leq n/2 \ \}$$

is a complete list of isomorphic indecomposable non free Cohen-Macaulay R-modules. On the other hand, if n is an odd integer, then

$$\{ \ I_i = (x^i, y) \mid 1 \leq i \leq (n-1)/2 \ \} \cup \{ \ N_+ = R/(x^{(n+1)/2} + \sqrt{-1}y), \ N_- = R/(x^{(n+1)/2} - \sqrt{-1}y) \ \}$$

is a complete list of the ones (cf. [8, Paragraph (9.9)]).

In this section, we shall show

Theorem 15. [3, Theorem 4.6] *Let $R = k[[x,y]]/(x^{n+1} + y^2)$. Then the stable hom order coincides with the stable degeneration order. Particularly, we have the following.*

(1) *If n is an even integer,*

$$\underline{0} \leq_{st} \underline{I_1} \leq_{st} \underline{I_2} \leq_{st} \cdots \leq_{st} \underline{I_{n/2}}.$$

(2) *If n is an odd integer,*

$$\underline{0} \leq_{st} \underline{I_1} \leq_{st} \underline{I_2} \leq_{st} \cdots \leq_{st} \underline{I_{(n-1)/2}} \leq_{st} \underline{N_+} \oplus \underline{N_-}.$$

and

$$\underline{N_\pm} \leq_{st} \underline{N_\pm} \oplus \underline{I_1} \leq_{st} \cdots \leq_{st} \underline{N_\pm} \oplus \underline{I_{(n-1)/2}} \leq_{st} \underline{N_\pm} \oplus \underline{N_+} \oplus \underline{N_-} \quad \textit{(double sign corresponds).}$$

To show this, we use the stable analogue of the argument over finite dimensional algebras in [11] The lemma below is well known for the case in an Abelian category (cf. [11, Lemma 2.6]). The same statement follows in an arbitrary k-linear triangulated category.

Lemma 16. [3, Lemma 4. 7] *Let*

$$\Sigma_1 : N_1 \xrightarrow{\binom{f_1}{v}} L_1 \oplus N_2 \xrightarrow{(u,g_1)} L_2 \longrightarrow N_1[1]$$

and

$$\Sigma_2 : M_1 \xrightarrow{\binom{f_2}{w}} N_1 \oplus M_2 \xrightarrow{(v,g_2)} N_2 \longrightarrow M_1[1]$$

be triangles in a k-linear triangulated category. Then we also have the following triangle.

$$M_1 \to L_1 \oplus M_2 \to L_2 \to M_1[1].$$

Definition 17. Let M and N be Cohen-Macaulay R-modules. We define a function $\delta_{\underline{M},\underline{N}}(-)$ on $\underline{\mathrm{CM}}(R)$ by

$$\delta_{\underline{M},\underline{N}}(-) = [-,\underline{N}] - [-,\underline{M}].$$

For a triangle $\underline{\Sigma} : \underline{L} \to \underline{M} \to \underline{N} \to \underline{L}[1]$, we also define a function $\delta_{\underline{\Sigma}}(-)$ on $\underline{\mathrm{CM}}(R)$ by

$$\delta_{\underline{\Sigma}}(-) = [-,\underline{L}] + [-,\underline{N}] - [-,\underline{M}].$$

Instead of giving the proof of Theorem 15, we give the concrete construction of the stable degenerations.

Example 18. Let $R = k[[x,y]]/(x^7 + y^2)$. Then $\underline{0} \leq_{st} \underline{I_1} \leq_{st} \underline{I_2} \leq_{st} \underline{I_3}$.

Proof. We show $\underline{I_1} \leq_{st} \underline{I_2}$. We construct the triangle Σ such that

$$\delta_\Sigma = \delta_{\underline{I_1},\underline{I_2}}.$$

(This Σ induces the triangle which gives the stable degenerations.)

Note that the table of dimension of $\underline{\mathrm{Hom}}_R$ and AR triangles on $\underline{\mathrm{CM}}(R)$ are the following:

$[-,-]$	$\underline{I_1}$	$\underline{I_2}$	$\underline{I_3}$
$\underline{I_1}$	2	2	2
$\underline{I_2}$	2	4	4
$\underline{I_3}$	2	4	6

$$\begin{aligned} \Sigma_{\underline{I_1}} &: \underline{I_1} \to \underline{I_2} \to \underline{I_1}, \\ \Sigma_{\underline{I_2}} &: \underline{I_2} \to \underline{I_1} \oplus \underline{I_3} \to \underline{I_2}, \\ \Sigma_{\underline{I_3}} &: \underline{I_3} \to \underline{I_2} \oplus \underline{I_3} \to \underline{I_3}. \end{aligned}$$

We also note that $\delta_{\Sigma_{\underline{I_i}}}(\underline{X}) = 2$ if $\underline{X} = \underline{I_i}$, or 0 if not.

First, since $\delta_{\underline{I_1},\underline{I_2}}(\underline{X}) = \begin{cases} 0 & \text{if } \underline{X} = \underline{I_1}, \\ 2 & \text{if } \underline{X} = \underline{I_2}, \\ 2 & \text{if } \underline{X} = \underline{I_3} \end{cases}$ and $\mu(\underline{I_2},\underline{I_i}) = 0$ if $i \neq 2$, we take the AR triangle $\Sigma_{\underline{I_2}} : \underline{I_2} \to \underline{I_1} \oplus \underline{I_3} \to \underline{I_2}$. Then we see that $\delta_{\Sigma_{\underline{I_2}}} < \delta_{\underline{I_1},\underline{I_2}}$.

Now $\delta_{\Sigma_{\underline{I_2}}}(\underline{I_3}) = 0$ but $\delta_{\underline{I_1},\underline{I_2}}(\underline{I_3}) = 2$. Apply Lemma 16 to $\Sigma_{\underline{I_2}}$ and $\Sigma_{\underline{I_3}}$, We obtain the triangle $\Sigma : \underline{I_3} \to \underline{I_1} \oplus \underline{I_3} \to \underline{I_2}$ such that $\delta_\Sigma = \delta_{\Sigma_{\underline{I_2}}} + \delta_{\Sigma_{\underline{I_3}}} = \delta_{\underline{I_1},\underline{I_2}}$. Therefore $\underline{I_1} \leq_{st} \underline{I_2}$. $\square$

Remark 19. By applying Lemma 16 to $\Sigma_{\underline{I_1}}$, $\Sigma_{\underline{I_2}}$ and $\Sigma_{\underline{I_3}}$, we have the diagram below:

$$\begin{array}{ccccccc} \underline{I_3} & \longrightarrow & \underline{I_2} & \longrightarrow & \underline{I_1} & \longrightarrow & \underline{0} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \underline{I_3} & \longrightarrow & \underline{I_3} & \longrightarrow & \underline{I_2} & \longrightarrow & \underline{I_1}. \end{array}$$

Thus we obtain $\underline{0} \leq_{st} \underline{I_1} \leq_{st} \underline{I_2} \leq_{st} \underline{I_3}$.

Now we consider the following condition which is the necessary condition to make the stable hom order a partial order over hypersurface rings:

- $(*)$ For an AR triangle $\underline{Z} \to \underline{Y} \to \underline{X} \to \underline{Z}[1]$, $[\underline{X}] + [\underline{Z}] - [\underline{Y}] = [\underline{X}[-1]] + [\underline{Z}[-1]] - [\underline{Y}[-1]]$ in $\mathrm{G}(\underline{\mathrm{CM}}(R))$.

More generally, on Theorem 15, we have the following result.

Theorem 20. [11, Paragraph (3.3)][3, Theorem 4.12] *Let R be a simple hypersurface singularity which satisfies $(*)$. Then $\underline{M} \leq_{hom} \underline{N}$ if and only if $\underline{M} \leq_{st} \underline{N}$ for Cohen-Macaulay R-modules M and N with $[\underline{M}] = [\underline{N}]$ in $K_0(\underline{\mathrm{CM}}(R))$.*

The following lemma is known as the Knörrer's periodicity (cf. [8, Theorem 12.10]).

Lemma 21. *Let $S = k[[x_0, x_1, \cdots, x_n]]$ be a formal power series ring. For a non-zero element $f \in (x_0, x_1, \cdots, x_n)S$, we consider the two rings $R = S/(f)$ and $R^\sharp = S[[y, z]]/(f + y^2 + z^2)$. Then the stable categories $\underline{\mathrm{CM}}(R)$ and $\underline{\mathrm{CM}}(R^\sharp)$ are equivalent as triangulated categories.*

At the end of this note, we state the result on the stable degenerations over simple hypersurface singularities of type (A_n) of arbitrary dimension.

Theorem 22. *Let R be a simple hypersurface singularities of type (A_n). The following statements hold for Cohen-Macaulay R-modules M and N with $[\underline{M}] = [\underline{N}]$ in $K_0(\underline{\mathrm{CM}}(R))$.*

(1) *If R is of odd dimension then $\underline{M} \le_{st} \underline{N}$ if and only if $\underline{M} \le_{\underline{hom}} \underline{N}$.*
(2) *If R is of even dimension then $\underline{M} \le_{st} \underline{N}$ if and only if $\underline{M} \le_{tri} \underline{N}$.*

Proof. Since stable degenerations are preserved by stable categorical equivalences (cf. [10, Corollary 6.6]), by virtue of Knörrer's periodicity (Lemma 21), we have only to deal with the case $\dim R = 1$ to show (1) and the case $\dim R = 0$ to show (2). In the case of dimension 0, we know the stable degeneration order coincides with the triangle order on $\underline{\mathrm{CM}}(R)$ (see [4, Corollary 2.12., Proposition 3.10.]). Hence, by Theorem 15, we obtain the assertion. □

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