CROSSED PRODUCTS FOR MATRIX RINGS

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Abstract. Let R be a ring and $n \geq 2$ an integer. We provide a systematic way to define new multiplications on $\mathrm{M}_n(R)$, the ring of $n \times n$ full matrices with entries in R. The obtained new rings Λ are Auslander-Gorenstein if and only if so is R.

Introduction

Auslander-Gorenstein rings (see Definition 2) appear in various fields of current research in mathematics. For instance, regular 3-dimensional algebras of type A in the sense of Artin and Schelter, Weyl algebras over fields of characteristic zero, enveloping algebras of finite dimensional Lie algebras and Sklyanin algebras are Auslander-Gorenstein rings (see [2], [3], [4] and [12], respectively). The class of Auslander-Gorenstein rings contains two important particular classes of rings. One is the class of quasi-Frobenius rings and the other is the class of commutative Gorenstein rings. In [1, Section 3] and [9, Section 4] we have provided various constructions of Auslander-Gorenstein rings. In this note, we will provide a systematic construction of Auslander-Gorenstein rings starting from an arbitrary Auslander-Gorenstein ring.

Let us recall the notion of Frobenius extensions of rings due to Nakayama and Tsuzuku [10, 11] which we modify as follows (cf. [1, Section 1]). We use the notation A/R to denote that a ring A contains a ring R as a subring. We say that A/R is a Frobenius extension if the following conditions are satisfied: (F1) A is finitely generated as a left R-module; (F2) A is finitely generated projective as a right R-module; (F3) there exists an isomorphism $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ in Mod-A. Note that ϕ induces a unique ring homomorphism $\theta : R \to A$ such that $x\phi(1) = \phi(1)\theta(x)$ for all $x \in R$. A Frobenius extension A/R is said to be of first kind if $A \cong \mathrm{Hom}_R(A, R)$ as R-A-bimodules, and to be of second kind if there exists an isomorphism $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ in Mod-A such that the associated ring homomorphism $\theta : R \to A$ induces a ring automorphism of R. Note that a Frobenius extension of first kind is a special case of a Frobenius extension of second kind. Let A/R be a Frobenius extension. Then A is an Auslander-Gorenstein ring if so is R, and the converse holds true if A is projective as a left R-module, and if A/R is split, i.e., the inclusion $R \to A$ is a split monomorphism of R-R-bimodules. It should be noted that A is projective as a left R-module if A/R is of second kind.

Fix a set of integers $I = \{0, 1, \ldots, n-1\}$ with $n \geq 2$ arbitrary. To begin with, starting from an arbitrary ring R, we will construct an I-graded ring A so that A/R is a split Frobenius extension of second kind. Namely, we will define an appropriate multiplication on a free right R-module A with a basis $\{u_i\}_{i \in I}$ using the following two data: a certain pair (q, χ) of an integer q and a mapping $\chi : \mathbb{Z}_+ \to \mathbb{Z}$; a certain triple (σ, c, t) of $\sigma \in \mathrm{Aut}(R)$

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and $c, t \in R^\sigma$, the fixed subring of R under σ. Then we will define an appropriate multiplication on a free right A-module Λ with a basis $\{v_i\}_{i \in I}$ so that Λ/A is a Frobenius extension of first kind. To do so, we need the group structure of I. Since we have to distinguish the addition in I and that in $\mathbb{Z}_+$, we fix a cyclic permutation of I

$$\pi = \begin{pmatrix} 0 & 1 & \cdots & n-1 \\ 1 & 2 & \cdots & 0 \end{pmatrix}$$

and make I a cyclic group with 0 the unit element by the law of composition $I \times I \to I, (i, j) \mapsto \pi^j(i)$. Then, as a right R-module, Λ has a basis $\{e_{ij}\}_{i,j \in I}$ such that $e_{ij}e_{kl} = 0$ unless $j = k$, $e_{ij}e_{jk} = e_{ik}c_{ijk}$ with $c_{ijk} \in R$ for all $i, j, k \in R$ and $xe_{ij} = e_{ij}\sigma_{ij}(x)$ with $\sigma_{ij} \in \mathrm{Aut}(R)$ for all $x \in R$ and $i, j \in I$. Using the above two data, we will provide a concrete construction of such families $\{c_{ijk}\}_{i,j,k \in I}$ and $\{\sigma_{ij}\}_{i,j \in I}$. However, except very simple cases, it would be a rather hard task to find out such families independently. This is the reason why we divide the construction into two steps.

1. Preliminaries

For a ring R we denote by $R^\times$ the set of units in R, by $\mathrm{Z}(R)$ the center of R, by $\mathrm{Aut}(R)$ the group of ring automorphisms of R, and for $\sigma \in \mathrm{Aut}(R)$ by R^σ the subring of R consisting of all $x \in R$ with $\sigma(x) = x$. The identity element of a ring is simply denoted by 1. We denote by Mod-R the category of right R-modules. Left R-modules are considered as right R^{op}-modules, where R^{op} denotes the opposite ring of R. In particular, we denote by inj dim R (resp., inj dim R^{op}) the injective dimension of R as a right (resp., left) R-module and by $\mathrm{Hom}_R(-,-)$ (resp., $\mathrm{Hom}_{R^{\mathrm{op}}}(-,-)$) the set of homomorphisms in Mod-R (resp., Mod-R^{op}).

We start by recalling the notion of Auslander-Gorenstein rings.

Proposition 1 (Auslander)**.** *Let R be a right and left noetherian ring. Then for any $n \geq 0$ the following are equivalent.*

(1) *In a minimal injective resolution $I^\bullet$ of R in* Mod-R*,* flat dim $I^i \leq i$ *for all* $0 \leq i \leq n$.
(2) *In a minimal injective resolution $J^\bullet$ of R in* Mod-R^{op}*,* flat dim $J^i \leq i$ *for all* $0 \leq i \leq n$.
(3) *For any $1 \leq i \leq n+1$, any $M \in$* mod-R *and any submodule X of* $\mathrm{Ext}^i_R(M, R) \in$ mod-R^{op} *we have* $\mathrm{Ext}^j_{R^{\mathrm{op}}}(X, R) = 0$ *for all* $0 \leq j < i$.
(4) *For any $1 \leq i \leq n+1$, any $X \in$* mod-R^{op} *and any submodule M of* $\mathrm{Ext}^i_{R^{\mathrm{op}}}(X, R) \in$ mod-R *we have* $\mathrm{Ext}^j_R(M, R) = 0$ *for all* $0 \leq j < i$.

Definition 2 ([4])**.** A right and left noetherian ring R is said to satisfy the Auslander condition if it satisfies the equivalent conditions in Proposition 1 for all $n \geq 0$, and to be an Auslander-Gorenstein ring if it satisfies the Auslander condition and inj dim $R =$ inj dim $R^{\mathrm{op}} < \infty$.

It should be noted that for a right and left noetherian ring R we have inj dim $R =$ inj dim R^{op} whenever inj dim $R < \infty$ and inj dim $R^{\mathrm{op}} < \infty$ (see [13, Lemma A]).

Next, we recall the notion of Frobenius extensions of rings due to Nakayama and Tsuzuku [10, 11], which we modify as follows (cf. [1, 7]).

Definition 3 ([7]). A ring A is said to be an extension of a ring R if A contains R as a subring, and the notation A/R is used to denote that A is an extension ring of R. A ring extension A/R is said to be Frobenius if the following conditions are satisfied:

(F1) A is finitely generated as a left R-module;
(F2) A is finitely generated projective as a right R-module;
(F3) $A \cong \mathrm{Hom}_R(A, R)$ as right A-modules.

Proposition 4 ([7, Proposition 1.4]). *Let A/R be a ring extension and $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ an isomorphism in* Mod-A. *Then the following hold.*

(1) *There exists a unique ring homomorphism $\theta : R \to A$ such that $x\phi(1) = \phi(1)\theta(x)$ for all $x \in R$.*
(2) *If $\phi' : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ is another isomorphism in* Mod-A, *then there exists $u \in A^\times$ such that $\phi'(1) = \phi(1)u$ and $\theta'(x) = u^{-1}\theta(x)u$ for all $x \in R$.*
(3) *ϕ is an isomorphism of R-A-bimodules if and only if $\theta(x) = x$ for all $x \in R$.*

Definition 5 (cf. [10, 11]). A Frobenius extension A/R is said to be of first kind if $A \cong \mathrm{Hom}_R(A, R)$ as R-A-bimodules, and to be of second kind if there exists an isomorphism $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ in Mod-A such that the associated ring homomorphism $\theta : R \to A$ induces a ring automorphism $\theta : R \xrightarrow{\sim} R$.

Proposition 6 ([7, Proposition 1.6]). *If A/R is a Frobenius extension of second kind, then A is projective as a left R-module.*

Proposition 7 ([7, Proposition 1.7]). *For any Frobenius extensions Λ/A, A/R the following hold.*

(1) *Λ/R is a Frobenius extension.*
(2) *Assume Λ/A is of first kind. If A/R is of second (resp., first) kind, then so is Λ/R.*

Definition 8 ([1]). A ring extension A/R is said to be split if the inclusion $R \to A$ is a split monomorphism of R-R-bimodules.

Proposition 9 ([7, Proposition 1.9]). *For any Frobenius extension A/R the following hold.*

(1) *If R is an Auslander-Gorenstein ring, then so is A with* $\mathrm{inj\ dim}\ A \le \mathrm{inj\ dim}\ R$.
(2) *Assume A is projective as a left R-module and A/R is split. If A is an Auslander-Gorenstein ring, then so is R with* $\mathrm{inj\ dim}\ R = \mathrm{inj\ dim}\ A$.

2. Construction

Throughout the rest of this note, we fix a ring R and an integer $n \ge 2$. We will provide a systematic way to define new multiplications on $\mathrm{M}_n(R)$, the ring of $n \times n$ full matrices with entries in R (cf. Theorem 15 below). However, we divide the construction into two steps, i.e., we will construct two ring extensions A/R and Λ/A such that $\Lambda \cong \mathrm{M}_n(R)$ as right R-modules. Except very simple cases, the direct construction would be a rather hard task (cf. [1] and [6]).

We need the following two data. Let $I = \{0, 1, \ldots, n-1\}$ be a set of integers and $\mathbb{Z}_+$ the set of non-negative integers. We fix a pair (q, χ) of an integer $q \in \mathbb{Z}$ and a mapping $\chi : \mathbb{Z}_+ \to \mathbb{Z}$ satisfying the following conditions:

(X0) $\chi(0) = 0$;

(X1) $\chi(i + kn) = \chi(i) + kq$ for all $(i, k) \in I \times \mathbb{Z}_+$;

(X2) $\chi(i) + \chi(j) \geq \chi(i + j)$ for all $i, j \in \mathbb{Z}_+$.

Also, we fix a triple (σ, c, t) of $\sigma \in \mathrm{Aut}(R)$ and $c, t \in R^\sigma$ satisfying the following condition:

$$(*) \quad xc = c\sigma(x), xt = t\sigma^q(x) \text{ for all } x \in R.$$

It should be noted that $ct = tc$.

Example 10. For any pair of integers (p, q) with $np \geq q$, setting

$$\chi(i + kn) = ip + kq$$

for $(i, k) \in I \times \mathbb{Z}_+$, we have a pair (q, χ) satisfying (X0), (X1) and (X2) and, setting

$$\varrho = \sigma^p \in \mathrm{Aut}(R) \quad \text{and} \quad s = tc^{np-q} \in R,$$

we have $s \in R^\varrho$ and $xs = s\varrho^n(x)$ for all $x \in R$.

Example 11. For any $c \in R^\times$ and $s \in \mathrm{Z}(R)$, setting $\sigma(x) = c^{-1}xc$ for $x \in R$ and $t = sc^q$ with $q \in \mathbb{Z}$ arbitrary, we have a triple (σ, c, t) of $\sigma \in \mathrm{Aut}(R)$ and $c, t \in R^\sigma$ satisfying the condition (*).

At first, we will construct a split ring extension A/R. Let A be a free right R-module with a basis $\{u_i\}_{i \in I}$. We set

$$u_{i+kn} = u_i t^k$$

for $(i, k) \in I \times \mathbb{Z}_+$ and

$$\omega(i, j) = \chi(i) + \chi(j) - \chi(i + j)$$

for $i, j \in \mathbb{Z}_+$. Note that ω is symmetric, i.e., $\omega(i, j) = \omega(j, i)$ for all $i, j \in \mathbb{Z}_+$ and that $\omega(i + kn, j + ln) = \omega(i, j)$ for all $(i, k), (j, l) \in I \times \mathbb{Z}_+$. Since by (X2) $\omega(i, j) \geq 0$ for all $i, j \in \mathbb{Z}_+$, we can define a multiplication on A subject to the following axioms:

(A1) $u_i u_j = u_{i+j} c^{\omega(i,j)}$ for all $i, j \in \mathbb{Z}_+$;

(A2) $xu_i = u_i \sigma^{\chi(i)}(x)$ for all $x \in R$ and $i \in \mathbb{Z}_+$,

where as usual we require $a^0 = 1$ for $a \in R$ even if $a = 0$. We denote by $\{\delta_i\}_{i \in I}$ the dual basis of $\{u_i\}_{i \in I}$ for the free left R-module $\mathrm{Hom}_R(A, R)$, i.e., $a = \sum_{i \in I} u_i \delta_i(a)$ for all $a \in A$. Then for any $a, b \in A$ we have

$$ab = \sum_{i,j \in I} u_{i+j} c^{\omega(i,j)} \sigma^{\chi(j)}(\delta_i(a)) \delta_j(b).$$

Proposition 12 ([8, Proposition 2.3(1)]). *A is an associative ring with $1 = u_0$ and contains R as a subring via the injective ring homomorphism $R \to A, x \mapsto u_0 x$.*

Example 13. Consider the case where $q = n - 1$ and $\chi(i + kn) = i + kq$ for all $(i, k) \in I \times \mathbb{Z}_+$. Then $\omega(i, j) = 0$ if $i + j < n$ and $\omega(i, j) = 1$ otherwise for $i, j \in I$. Also, $x(tc) = (tc)\sigma^n(x)$ for all $x \in R$. Let $R[X; \sigma]$ be a right skew polynomial ring with trivial derivation, i.e., $R[X; \sigma]$ consists of all polynomials in an indeterminate X with the right-hand coefficients in R and the multiplication is defined subject to the following rule: $xX = X\sigma(x)$ for all $x \in R$. It then follows that $(X^n - tc) = (X^n - tc)R[X;\sigma]$ is a two-sided ideal of $R[X;\sigma]$ and $A \cong R[X;\sigma]/(X^n - tc)$ as extension rings of R.

Next, we will specialize the construction given in [7, Section 2] and construct a ring extension Λ/A. To do so, we need the group structure of I. In order to distinguish the addition in I and that in $\mathbb{Z}_+$, we fix a cyclic permutation of I

$$\pi = \begin{pmatrix} 0 & 1 & \cdots & n-1 \\ 1 & 2 & \cdots & 0 \end{pmatrix}$$

and make I a cyclic group with 0 the unit element by the law of composition $I \times I \to I, (i,j) \mapsto \pi^j(i)$. It should be noted that

$$i + j = \begin{cases} \pi^j(i) & \text{if } i + j < n, \\ \pi^j(i) + n & \text{if } i + j \geq n \end{cases}$$

for all $i, j \in I$. Thus, setting

$$\epsilon(i,j) = \begin{cases} 0 & \text{if } i + j < n, \\ 1 & \text{if } i + j \geq n \end{cases}$$

for $i, j \in I$, we have $i + j = \pi^j(i) + \epsilon(i,j)n$ and hence

$$\chi(i+j) = \chi(\pi^j(i)) + \epsilon(i,j)q \quad \text{and} \quad u_{i+j} = u_{\pi^j(i)}t^{\epsilon(i,j)}$$

for all $i, j \in I$.

Setting $A_i = u_i R$ for $i \in I$, $A = \oplus_{i \in I} A_i$ yields an I-graded ring with $A_0 = R$. Note however that in the above A is constructed as a residue ring of a positively graded ring (cf. Example 13 above).

We denote by $\varepsilon_i : A \to A_i, a \mapsto u_i\delta_i(a)$ the projection for each $i \in I$. Then the following conditions are satisfied:

(E1) $\varepsilon_i\varepsilon_j = 0$ unless $i = j$ and $\sum_{i \in I} \varepsilon_i = \mathrm{id}_A$;

(E2) $\varepsilon_i(a)\varepsilon_j(b) = \varepsilon_{\pi^j(i)}(\varepsilon_i(a)b)$ for all $a, b \in A$ and $i, j \in I$.

Let Λ be a free right A-module with a basis $\{v_i\}_{i \in I}$ and define a multiplication on Λ subject to the following axioms:

(L1) $v_iv_j = 0$ unless $i = j$ and $v_i^2 = v_i$ for all $i \in I$;

(L2) $av_i = \sum_{j \in I} v_j\varepsilon_{\pi^{-i}(j)}(a)$ for all $a \in A$ and $i \in I$.

Let us denote by $\{\gamma_i\}_{i \in I}$ the dual basis of $\{v_i\}_{i \in I}$ for the free left A-module $\mathrm{Hom}_A(\Lambda, A)$, i.e., $\lambda = \sum_{i \in I} v_i\gamma_i(\lambda)$ for all $\lambda \in \Lambda$. It is not difficult to see that

$$\lambda\mu = \sum_{i,j \in I} v_i\varepsilon_{\pi^{-j}(i)}(\gamma_i(\lambda))\gamma_j(\mu)$$

for all $\lambda, \mu \in \Lambda$.

Proposition 14. *Λ is an associative ring with $1 = \sum_{i \in I} v_i$ and contains A as a subring via the injective ring homomorphism $A \to \Lambda, a \mapsto \sum_{i \in I} v_i a$.*

Note that $\{v_iu_j\}_{i,j \in I}$ is a basis for the free right R-module Λ with $\{\delta_j\gamma_i\}_{i,j \in I}$ the dual basis for the free left R-module $\mathrm{Hom}_R(\Lambda, R)$, and that for any $i \in I$ by (L2) we have $xv_i = v_ix$ for all $x \in R$ and hence Λv_i is a Λ-R-bimodule. Similarly, every $v_i\Lambda$ is an

R-Λ-bimodule. Also, by (L2) $u_k v_j = v_{\pi^k(j)} u_k$ for all $j, k \in I$, so that $v_i \Lambda v_j = v_i u_{\pi^{-j}(i)} R$ and

$$\mathrm{Hom}_\Lambda(v_j \Lambda, v_i \Lambda) \xrightarrow{\sim} R, f \mapsto \delta_{\pi^{-j}(i)}(\gamma_i(f(v_j)))$$

as R-R-bimodules for all $i, j \in I$. In particular,

$$\mathrm{End}_\Lambda(v_i \Lambda) \xrightarrow{\sim} R, f \mapsto \delta_0(\gamma_i(f(v_i)))$$

as rings for all $i \in I$.

Now, setting $e_{ij} = v_i u_{\pi^{-j}(i)}$ for $i, j \in I$, we have a basis $\{e_{ij}\}_{i,j \in I}$ for the free right R-module Λ. Then, as remarked above, we have $v_i \Lambda v_j = e_{ij} R$ for all $i, j \in I$ and $\{\delta_{\pi^{-j}(i)} \gamma_i\}_{i,j \in I}$ is the duel basis of $\{e_{ij}\}_{i,j \in I}$ for the free left R-module $\mathrm{Hom}_R(\Lambda, R)$, i.e.,

$$\lambda = \sum_{i,j \in I} e_{ij} \delta_{\pi^{-j}(i)}(\gamma_i(\lambda))$$

for all $\lambda \in \Lambda$. In particular,

$$\rho : \Lambda \xrightarrow{\sim} \mathrm{M}_n(R), \lambda \mapsto (\delta_{\pi^{-j}(i)}(\gamma_i(\lambda)))_{i,j \in I}$$

as right R-modules.

Theorem 15. *The multiplication in Λ is subject to the following axioms:*

(M1) $e_{ij} e_{kl} = 0$ unless $j = k$;

(M2) $e_{ij} e_{jk} = e_{ik} t^{\epsilon(\pi^{-j}(i), \pi^{-k}(j))} c^{\omega(\pi^{-j}(i), \pi^{-k}(j))}$ for all $i, j, k \in I$;

(M3) $x e_{ij} = e_{ij} \sigma^{\chi(\pi^{-j}(i))}(x)$ for all $x \in R$ and $i, j \in I$.

It should be noted that the axioms above induce another multiplication on $\mathrm{M}_n(R)$ such that ρ is a ring isomorphism.

In the following, setting

$$\Delta_k = \{(i, j) \in I \times I \mid \pi^{-j}(i) = k\}$$

for $k \in I$, we decompose $I \times I$ into a disjoint unioun $I \times I = \cup_{k \in I} \Delta_k$. Note that the Δ_k are π-stable, i.e., $(\pi(i), \pi(j)) \in \Delta_k$ for all $k \in I$ and $(i, j) \in \Delta_k$.

Lemma 16. *We have $v_k = e_{kk}$ and $u_k = \sum_{(i,j) \in \Delta_k} e_{ij}$ for all $k \in I$.*

It follows by Lemma 16 that the axioms (M1), (M2) and (M3) imply the axioms (A1), (A2), (L1) and (L2). Unfortunately, it would be rather hard to check directly that the multiplication defined on Λ subject to the axioms (M1), (M2) and (M3) is actually associative. This is the reason why we divided the construction into two steps. However, we notice the following which could be of use for a direct verification.

Lemma 17. *For any $i, j, k \in I$ the following hold.*

(1) $\epsilon(\pi^{-j}(i), \pi^{-k}(j)) = 0$ *if and only if one of the following cases occurs: $i < k < j$, $j < i < k$, $k < j < i$, $i = j$ or $j = k$.*

(2) $\epsilon(\pi^{-j}(i), \pi^{-k}(j)) = 1$ *if and only if one of the following cases occurs: $i < j < k$, $k < i < j$, $j < k < i$ or $i = k \neq j$.*

(3) $\chi(\pi^{-j}(i) + \pi^{-k}(j)) = \chi(\pi^{-k}(i)) + \epsilon(\pi^{-j}(i), \pi^{-k}(j)) q$.

Example 18. Let (q, χ) and (ϱ, s) be as in Example 10. Then we have $e_{ij} e_{jk} = e_{ik} s^{\epsilon(\pi^{-j}(i), \pi^{-k}(j))}$ for all $i, j, k \in I$ and $x e_{ij} = e_{ij} \varrho^{\pi^{-j}(i)}(x)$ for all $x \in R$ and $i, j \in I$.

According to Theorem 15, it is easy to see that there exists $\eta \in \mathrm{Aut}(\Lambda)$ such that $\eta(e_{ij}) = e_{\pi(i),\pi(j)}$ for all $i, j \in I$ and $\eta(x) = x$ for all $x \in R$.

Proposition 19. *We have* $\Lambda^\eta = A$.

Definition 20. Let J be a non-empty subset of I and $\Lambda_J = \oplus_{i,j\in J} e_{ij}R$. We define a multiplication on Λ_J subject to the axioms (M1), (M2) and (M3). Then $\mathrm{End}_\Lambda(\oplus_{i\in J} v_i\Lambda) \cong \oplus_{i,j\in J} v_i\Lambda v_j = \Lambda_J$ as rings.

3. Frobenius extensions

In this section, we will study the structure of the ring extension Λ/R.

In the following, we set $\gamma = \sum_{i\in I} \gamma_i \in \mathrm{Hom}_A(\Lambda, A)$.

Proposition 21. *There exists an isomorphism* $\psi : \Lambda \xrightarrow{\sim} \mathrm{Hom}_A(\Lambda, A), \lambda \mapsto \gamma\lambda$ *of* A-Λ-*bimodules, i.e.,* Λ/A *is a Frobenius extension of first kind.*

Proposition 22 ([7, Proposition 2.3(2)])**.** *There exists an isomorphism* $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R), a \mapsto \delta_{n-1}a$ *in* Mod-A *such that* $x\phi(1) = \phi(1)\sigma^{-\chi(n-1)}(x)$ *for all* $x \in R$*, i.e.,* A/R *is a split Frobenius extension of second kind.*

Corollary 23. *A is an Auslander-Gorenstein ring if and only if so is R.*

Theorem 24. *The following hold.*

(1) $v_i\Lambda \xrightarrow{\sim} \mathrm{Hom}_R(\Lambda v_{\pi(i)}, R), \lambda \mapsto \delta_{n-1}\gamma_i\lambda$ *in* Mod-Λ *for all* $i \in I$.
(2) Λ/R *is a split Frobenius extension of second kind.*
(3) Λ *is an Auslander-Gorenstein ring if and only if so is* R.

Remark 25. It follows by Corollary 23 and Theorem 24(3) that Λ is an Auslander-Gorenstein ring if and only if so is A. If $n \cdot 1 \in R^\times$, then Λ/A is split. But we do not know whether or not Λ/A is always split.

It follows by Propositions 21, 22 that we have an isomorphism in Mod-Λ

$$\xi : \Lambda \xrightarrow{\sim} \mathrm{Hom}_R(\Lambda, R), \lambda \mapsto \delta_{n-1}\gamma\lambda$$

such that $x\xi(1) = \xi(1)\sigma^{-\chi(n-1)}(x)$ for all $x \in R$.

Proposition 26. *We have* $\xi(e_{ij}) = \delta_{\pi^{-\pi(i)}(j)}\gamma_j$ *for all* $i, j \in I$.

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