ON THE HOCHSCHILD (CO)HOMOLOGY OF A MONOMIAL ALGEBRA GIVEN BY A CYCLIC QUIVER AND TWO ZERO-RELATIONS

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Abstract. Let K be an algebraically closed field and Γ_s a cyclic quiver with s vertices. Xu and Wang investigated the Hochschild (co)homology groups of $K\Gamma_s/I$, where I is an ideal of $K\Gamma_s$ generated by one path. In this paper, in the case that I is an ideal of $K\Gamma_s$ generated by two paths, we give the module structure of the Hochschild (co)homology of $K\Gamma_s/I$ and the necessary and sufficient condition for the Hochschild cohomology ring of $K\Gamma_s/I$ to be finitely generated.

1. Introduction

The Hochschild cohomology of algebras is one of important invariances of derived equivalence and it has various algebraic structures, for example, module structure, ring structure and Lie algebraic structure etc. However, in general, it is difficult to determine these structures of Hochschild cohomology.

For a monomial algebra over an algebralically closed field, Bardzell [1] gave its minimal projective bimodule resolution. Green and Snashall [3] constructed Bardzell's minimal projective resolution again by means of "overlap". By means of this minimal projective resolution, for some classes of monomial algebras, the module structure and ring structure of the Hochschild cohomology are investigated. Moreover, Green, Snashall and Solberg [4] determined the condition that the Hochschild cohomology ring modulo nilpotence of a monomial algebra is finitely generated. However, for a monomial algebra, even the module structure of the Hochschild cohomology is not completely determined.

While, Han [5] gave the Hochschild homology groups of a monomial algebra over a field by means of the Hochschild homology groups of bound quiver algebras given by cyclic subquivers of its ordinary quiver. So it is important to determine the module structure of the Hochschild homology of a bound quiver algebra of a cyclic quiver. We also think that it is important to compute the module structure of the Hochschild cohomology of a bound quiver algebra of a cyclic quiver.

Let K be an algebraically closed field, $s \geq 3$ a positive integer, Γ_s the cyclic quiver with s vertices and I an admissible ideal. Then it is easy to see that $K\Gamma_s/I$ is a monomial algebra. The cardinal number of the minimal set of paths in the generating set of I is equal to s if and only if $K\Gamma_s/I$ is a truncated cycle algebra, that is, $I = R_{\Gamma_s}^m$ for some integer m, where R_{Γ_s} is the arrow ideal of $K\Gamma_s$. In that case, the module structure of the Hochschild cohomology is determined in [2] and [10]. On the other hand, for an algebra $K\Gamma_s/I$ with an ideal I generated by only one path, Xu and Wang [9] investigated its

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Hochschild homology and cohomology. By the result in [6], for bound quiver algebras of a cyclic quiver, the module structure of the Hochschild homology is given by the Hochschild homology of truncated cycle algebras. In particular, the Hochschild homology of truncated cycle algebras is computed by Han [5] and Sköldberg [8]. However, the dimension formula of the Hochschild homology of bound quiver algebras of a cyclic quiver is not known completely.

In this paper, we will determine the module structure of the Hochschild (co)homology groups of $K\Gamma_s/I$ with an ideal I generated by two paths by means of Bardzell's minimal projective resolution treated in [3]. Let p_1 and p_2 be paths in Γ_s and I the ideal of $K\Gamma_s$ generated by p_1 and p_2. We deal with the bound quiver algebra $A := K\Gamma_s/I$. For each $i = 1, 2$, we denote the start points and end points of p_i by $o(p_i)$ and $t(p_i)$, respectively. Moreover, we put $o(p_1) = 0$, and we denote the length of p_i by $l(p_i)$ for $i = 1, 2$. Let N_i and r_i $(i = 1, 2)$ be positive integers which satisfy that $l(p_i) = N_i s + r_i$, $N_i \geq 0$ and $1 \leq r_i \leq s$ for each $i = 1, 2$. Furthermore, let $t_i (i = 1, 2)$ be the positive integers which satisfy that $t_i \equiv t(p_i) \pmod{s}$ and $1 \leq t_i \leq s$ for each $i = 1, 2$, and let $s_2 = o(p_2)$.

We divide into the following cases, and we construct the minimal projective resolution of A and determine the module structure of the Hochschild (co)homology of A for each cases:

- $N_1 = N_2 + 1$,
- $N_1 = N_2$
  Case 1: $0 < s_2 < t_1 < t_2 (\leq s)$,
  Case 2: $0 < t_2 \leq s_2 < t_1 (\leq s)$,
  Case 3: $0 < t_1 \leq s_2 < t_2 (\leq s)$.

Moreover, we also give the necessary and sufficient condition for the Hochschild cohomology ring $\mathrm{HH}^*(A)$ of A to be finitely generated for each cases.

Throughout this paper, we denote $\otimes_K$ by $\otimes$ for the sake of simplicity.

2. The Hochschild (co)homology groups of A

Let K be an algebraically closed field and s a positive integer satisfying $s \geq 3$ and Γ_s the cyclic quiver with s vertices $0, 1, \ldots, s-1$ and s arrows $\alpha_0, \alpha_1, \ldots, \alpha_{s-1}$ such that $o(\alpha_{s-1}) = s-1$, $t(\alpha_{s-1}) = 0$, $o(\alpha_i) = i$ and $t(\alpha_i) = i+1$ for $i = 0, 1, \ldots s-2$. We consider the path algebra $K\Gamma_s$ over K. It is satisfied that $\alpha_i = e_i \alpha_i e_{i+1}$ for each $0 \leq i \leq s-1$, where the subscripts i of e_i and α_i are considered to be modulo s. Let X be the sum of all arrows in KQ. Let p_1 and p_2 be paths in Γ_s and I an ideal of $K\Gamma_s$ generated by p_1 and p_2. In this section we compute the Hochschild (co) homology groups of the bound quiver algebra $A = K\Gamma_s/I$.

Throughout this paper, we assume that $o(p_1) = 0$. We set the integer t_i satisfying $t_i \equiv t(p_i) \pmod{s}$ and $1 \leq t_i \leq s$ for each $i = 1, 2$, and we also set the integer $s_2 = o(p_2)$.

Let $l(p_i)$ be the length of p_i for $i = 1, 2$. For each $i = 1, 2$ we set the integers N_i and r_i which satisfy $l(p_i) = N_i s + r_i$ and $1 \leq r_i \leq s$. It is clear that $|N_1 - N_2| \geq 2$ contradicts the definition of the minimal set of paths in the generating set of I. Therefore, we investigate the Hochschild homology of A in the following two cases:

(1) $N_1 = N_2 + 1$,
(2) $N_1 = N_2$.

2.1. **The Hochschild (co)homology groups of A in the case $N_1 = N_2 + 1$.** In this section, we compute the Hochschild (co)homology groups of A in the case $N_1 = N_2 + 1$. We set $N = N_2$ for the sake of simplicity. Then the inequality $0 < t_1 < t_2 \leq s_2 < s$ holds. In fact, if $s_2 < t_2$ holds, then the length of p_1 and p_2 are given by $l(p_1) = (N+1)s + t_1$ and $l(p_2) = Ns + t_2 - s_2$. Hence We have $l(p_1) - (l(p_2) + o(p_2)) = s + t_1 - t_2 > 0$. Thus p_2 is a subpath of p_1, which is contradiction.

Therefore, p_1 and p_2 are written by $p_1 = o(p_1)X^{(N+1)s+t_1}$ and $p_2 = o(p_2)X^{Ns+s-s_2+t_2}$, respectively.

If $N = 0$, then $\mathrm{gl.dim}\, A < \infty$. Hence we assume that $N \geq 1$. We construct the Bardzell's projective resolution of A.

Theorem 1 ([7, Theorem 3.2])**.** *We set $N = N_2 \geq 1$. In the case that $N_1 = N_2 + 1$, then the projective resolution of A is given by*

$$
\begin{aligned}
P^0 &= \coprod_{i=0}^{s-1} Ae_i \otimes e_i A, \qquad P^1 = \coprod_{i=0}^{s-1} Ae_i \otimes e_{i+1} A,\\
P^2 &= Ae_0 \otimes e_{t_1} A \oplus Ae_{s_2} \otimes e_{t_2} A,\\
P^3 &= Ae_0 \otimes e_{t_2} A \oplus Ae_{s_2} \otimes e_{t_1} A,\\
P^n &= Ae_{s_2} \otimes e_{t_2} A \quad (n \geq 4)
\end{aligned}
$$

and

$$
\begin{aligned}
&d^0 : \text{multiplication},\\
&d^1 : e_i \otimes e_{i+1} \mapsto e_i(1 \otimes X - X \otimes 1)e_{i+1},\\
&d^2 : e_0 \otimes e_{t_1} \mapsto e_0\Big(\sum_{k=0}^{(N+1)s+t_1-1} X^k \otimes X^{(N+1)s+t_1-k-1}\Big)e_{t_1},\\
&\quad e_{s_2} \otimes e_{t_2} \mapsto e_{s_2}\Big(\sum_{k=0}^{Ns+s-s_2+t_2-1} X^k \otimes X^{Ns+s-s_2+t_2-k-1}\Big)e_{t_2},\\
&d^3 : e_0 \otimes e_{t_2} \mapsto e_0(1 \otimes X^{t_2-t_1} - X_2^s \otimes 1)e_{t_2},\\
&\quad e_{s_2} \otimes e_{t_1} \mapsto e_{s_2}(1 \otimes X^{s-t_2+t_1} - X^{s-s_2} \otimes 1)e_{t_1},\\
&d^4 : e_{s_2} \otimes e_{t_2} \mapsto e_{s_2}\Big(\sum_{r=0}^{N}(X^{rs+s-s_2} \otimes X^{(N-r)s} + X^{rs} \otimes X^{(N-r)s+t_2-t_1}\Big)e_{t_2},\\
&d^{2c+1} : e_{s_2} \otimes e_{t_2} \mapsto e_{s_2}(1 \otimes X^s - X^s \otimes 1)e_{t_2},\\
&d^{2c+2} : e_{s_2} \otimes e_{t_2} \mapsto e_{s_2}\Big(\sum_{r=0}^{N} X^{rs} \otimes X^{(N-r)s}\Big)e_{t_2}
\end{aligned}
$$

for $c \geq 2$.

By the above projective resolution of A, we have the following results.

Theorem 2 ([7, Theorem 3.3])**.** *Suppose that $l(p_1) = (N+1)s + t_1$, $l(p_2) = Ns + t_2$ $(1 \leq t_1, t_2 \leq s)$. Then $0 < t_1 < t_2 \leq s_2 < s$ and the module structure of $\mathrm{HH}^n(A)$ over K*

is given by

$$\dim_K \mathrm{HH}^0(A) = \begin{cases} N+3 & \textit{if } t_1 = 1, \\ N+2 & \textit{if } t_1 \neq 1, \end{cases}$$
$$\dim_K \mathrm{HH}^1(A) = N+1,$$
$$\dim_K \mathrm{HH}^2(A) = \dim_K \mathrm{HH}^4(A) = \begin{cases} N & \textit{if } s_2 \neq t_2, \\ N+1 & \textit{if } s_2 = t_2, \end{cases}$$
$$\dim_K \mathrm{HH}^3(A) = N+1,$$
$$\dim_K \mathrm{HH}^n(A) = \begin{cases} N+1 & \textit{if } s_2 = t_2 \textit{ and } \operatorname{char} K \mid N+1, \\ N & \textit{otherwise} \end{cases}$$

for $n \geq 5$.

Theorem 3 ([7, Theorem 3.6])**.** *Suppose that* $l(p_1) = (N+1)s + t_1$, $l(p_2) = Ns + t_2$ $(1 \leq t_1, t_2 \leq s)$. *Then* $0 < t_1 < t_2 \leq o(p_2) < s$ *and the dimension of* $\mathrm{HH}_n(A)$ *over* K *is given by*

$$\dim_K \mathrm{HH}_0(A) = s + N,$$
$$\dim_K \mathrm{HH}_n(A) = \begin{cases} N+1 & \textit{if } \operatorname{char} K \mid N+1, \\ N & \textit{if } \operatorname{char} K \nmid N+1 \end{cases}$$

for $n \geq 1$.

2.2. **The Hochschild (co)homology of A in the case $N_1 = N_2$.** In this section, we compute the Hochschild (co)homology of A in the case $N_1 = N_2$. We set $N = N_2$ for the sake of simplicity. Then the following inequalities do not occur:

- $0 < s_2 < t_2 < t_1 \leq s$,
- $0 < t_1 < t_2 \leq s_2 < s$.

In fact, if one of the above condition holds, then p_2 is a subpath of p_1 or $|N_1 - N_2| = 1$, which contradicts the definition of p_1 and p_2. Moreover, the situations $0 < s_2 < t_1 < t_2 \leq s$ and $0 < t_2 < t_1 \leq s_2 < s$ are same situations.

Hence, we investigate the dimension of the Hochschild homology of A in the following three cases:

Case 1: $0 < s_2 < t_1 < t_2 \leq s$,
Case 2: $0 < t_2 \leq s_2 < t_1 \leq s$,
Case 3: $0 < t_1 \leq s_2 < t_2 \leq s$.

As is the case in the computation in subsection 2.1, by determining Bardzell's projective resolution, we can compute the module structure of the Hochschild cohomology and homology of A for each cases above.

Theorem 4 ([7, Theorem 4.3])**.** *Suppose that* $0 < s_2 < t_1 < t_2 \leq s$. *Then the module structure of the Hochschild cohomology of* A *over* K *is given by*

$$\dim_K \mathrm{HH}^0(A) = \begin{cases} N+2 & \textit{if } t_1 - s_2 \geq 2, \\ N+1 & \textit{otherwise}, \end{cases}$$
$$\dim_K \mathrm{HH}^1(A) = N+1,$$

$$\dim_K \mathrm{HH}^2(A) = N,$$

$$\dim_K \mathrm{HH}^3(A) = \begin{cases} N+1 & \textit{if } t_2 = s \textit{ and } \operatorname{char} K \mid N, \\ & \textit{or } \operatorname{char} K \mid N+1, \\ N & \textit{otherwise}, \end{cases}$$

$$\dim_K \mathrm{HH}^4(A) = \begin{cases} N+1 & \textit{if } \operatorname{char} K \mid N+1 \textit{ and } t_2 \neq s, \\ N & \textit{otherwise}, \end{cases}$$

$$\dim_K \mathrm{HH}^n(A) = \begin{cases} N+1 & \textit{if } \operatorname{char} K \mid N+1, \\ N & \textit{otherwise} \end{cases}$$

for $n \geq 5$.

Theorem 5 ([7, Theorem 4.6]). *Suppose that* $0 < s_2 < t_1 < t_2 \leq s$. *Then the dimension of* $\mathrm{HH}_n(A)$ *over* K *is given by*

$$\dim_K \mathrm{HH}_0(A) = s + N,$$

$$\dim_K \mathrm{HH}_n(A) = \begin{cases} N+1 & \textit{if } \operatorname{char} K \mid N+1, \\ N & \textit{if } \operatorname{char} K \nmid N+1 \end{cases}$$

for $n \geq 1$.

Theorem 6 ([7, Theorem 4.10]). *Suppose that* $l(p_1) = Ns + r_1$, $l(p_2) = Ns + r_2$ $(1 \leq r_1, r_2 \leq s)$. *Then* $0 < t_2 \leq s_2 < t_1 \leq s$ *and the module structure of* $\mathrm{HH}^n(A)$ *over* K *is given by*

$$\dim_K \mathrm{HH}^0(A) = \begin{cases} N+3 & \textit{if } t_1 - s_2 \geq 2 \textit{ and } t_2 \geq 2, \\ N+2 & \textit{otherwise}, \end{cases}$$

$$\dim_K \mathrm{HH}^1(A) = N+1,$$

$$\dim_K \mathrm{HH}^{2c}(A) = \begin{cases} N+1 & \textit{if } t_1 = s \textit{ and } t_2 = s_2, \\ N & \textit{otherwise}, \end{cases}$$

$$\dim_K \mathrm{HH}^{2c+1}(A) = \begin{cases} N+2 & \textit{if } t_1 \neq s \textit{ and } t_2 \neq s_2, \\ N+1 & \textit{otherwise} \end{cases}$$

for $c \geq 1$.

Theorem 7 ([7, Theorem 4.12]). *Suppose that* $0 < t_2 \leq s_2 < t_1 \leq s$. *Then the dimension of* $\mathrm{HH}_n(A)$ *over* K *is given by*

$$\dim_K \mathrm{HH}_n(A) = \begin{cases} s+N & \textit{if } n = 0, \\ N+1 & \textit{if } n > 0 \textit{ and } \operatorname{char} K \nmid N+1, \\ N+2 & \textit{if } n > 0 \textit{ and } \operatorname{char} K \mid N+1. \end{cases}$$

Theorem 8 ([7, Theorem 4.15]). *Suppose that* $0 < t_1 \leq s_2 < t_2 \leq s$. *Then the module structure of* $\mathrm{HH}^n(A)$ *is given by*

$$\dim_K \mathrm{HH}^0(A) = N+2,$$

$$\dim_K \mathrm{HH}^1(A) = N+1,$$

$$\dim_K \mathrm{HH}^{4c-2}(A) = N,$$

$$\dim_K \mathrm{HH}^{4c-1}(A) = \begin{cases} N+1 & \textit{if } t_1 = s_2,\ t_2 = s \textit{ and } \operatorname{char} K \mid 2N+1, \\ N & \textit{otherwise}, \end{cases}$$

$$\dim_K \mathrm{HH}^{4c}(A) = \begin{cases} N+1 & \textit{if } t_1 = s_2,\ t_2 = s \textit{ and } \operatorname{char} K \mid 2N+1, \\ & \textit{or } t_1 \neq s_2,\ t_2 \neq s, \\ N & \textit{otherwise}, \end{cases}$$

$$\dim_K \mathrm{HH}^{4c+1}(A) = \begin{cases} N & \textit{if } t_2 = s \textit{ or } t_1 = s_2, \\ N+1 & \textit{otherwise} \end{cases}$$

for $c \geq 1$.

Theorem 9 ([7, Theorem 4.16])**.** *Suppose that* $0 < t_1 \leq s_2 < t_2 \leq s$. *Then the dimension of* $\mathrm{HH}_n(A)$ *of* A *is given by*

$$\dim_K \mathrm{HH}_n(A) = \begin{cases} s+N & \textit{if } n = 0, \\ N+1 & \textit{if } \operatorname{char} k \mid 2N+1,\ n > 0 \textit{ and } n \equiv 0, 3 \pmod 4, \\ N & \textit{otherwise}. \end{cases}$$

3. The Hochschild cohomology ring of A

In this section, we give the necessary and sufficient condition for the Hochschild cohomology ring of A to be finitely generated.

We denote the Yoneda product in $\mathrm{HH}^*(A)$ by $\times$. Let $[\varphi] \in \mathrm{HH}^i(A)$ and $[\psi] \in \mathrm{HH}^j(A)$ be the elements which are represented by i-cocycle $\varphi \in \mathrm{Hom}_{A^e}(P_i, A)$ and j-cocycle $\psi \in \mathrm{Hom}_{A^e}(P_j, A)$, respectively. Then $[\varphi] \times [\psi] \in \mathrm{HH}^{i+j}(A)$ is given as follows: There exists the commutative diagram of A-bimodules

$$\begin{tikzcd} \cdots \arrow[r] & P_{i+j} \arrow[r, "d_{i+j}"] \arrow[d, "\sigma_i"'] & \cdots \arrow[r, "d_{j+2}"] & P_{j+1} \arrow[r, "d_{j+1}"] \arrow[d, "\sigma_1"] & P_j \arrow[r, "\psi"] \arrow[d, "\sigma_0"] & A \arrow[d, equal] & \\ \cdots \arrow[r] & P_i \arrow[r, "d_i"] & \cdots \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r, "d_0"] & A \arrow[r] & 0 \end{tikzcd}$$

where $\sigma_l (0 \leq l \leq i)$ are liftings of ψ. Then we have $[\varphi] \times [\psi] = [\varphi\sigma_i] \in \mathrm{HH}^{i+j}(A)$.

By giving generator of Hochschild cohomology groups and computing the Yoneda product in $\mathrm{HH}^*(A)$ for each cases, we have the following results.

Theorem 10. *Suppose that* $N_1 = N+1$ *and* $N_2 = N$. *Then the following are equivalent*:
(1) $\mathrm{HH}^*(A)$ *is finitely generated.*
(2) $N \geq 1$, $s_2 = t_2$ *and* $\operatorname{char} K \mid N+1$, *or* $N = 0$.

Theorem 11. *Suppose that* $N_1 = N_2 = N$ *and* $0 < s_2 < t_1 < t_2 \leq s$. *Then the following are equivalent*:
(1) $\mathrm{HH}^*(A)$ *is finitely generated.*
(2) $N = 0$.

Theorem 12. *Suppose that* $N_1 = N_2 = N$ *and* $0 < t_2 \leq s_2 \leq t_1 \leq s$. *Then the following are equivalent*:
(1) $\mathrm{HH}^*(A)$ *is finitely generated.*
(2) $t_1 = s$ *and* $s_2 = t_2$.

Theorem 13. *Suppose that* $N_1 = N_2 = N$ *and* $0 < t_1 \leq s_2 \leq t_2 \leq s$ *Then the following are equivalent*:

(1) $\mathrm{HH}^*(A)$ *is finitely generated.*

(2) $t_1 = s_2$ *and* $t_2 = s$*, or* $N = 0$.

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