ON MODULES OF INFINITE REDUCED GRADE

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Abstract. Let $R, A$ be right Noetherian rings and $V$ an $(A, R)$-bimodule. Our aim is to provide a sufficient condition on $V$ which enables $A$ to inherit from $R$ certain homological properties. Especially, we will show that if the generalized Nakayama conjecture is true for $R$ then so is for $A$.

Introduction

In this talk we are mainly concerned with the generalized Nakayama conjecture which states that if $R$ is a right Noetherian ring then every simple right $R$-module $S$ with $\text{Ext}^i_R(S, R) = 0$ for all $i \geq 1$ would be torsionless, i.e., $\text{Hom}_R(S, R) \neq 0$ (see [1]).

For any ring $R$, we denote by $\text{Mod}_R$ the category of right $R$-modules and by $\text{mod}_R$ the full subcategory of $\text{Mod}_R$ consisting of finitely presented modules, and left $R$-modules are considered as right $R^{\text{op}}$-modules, where $R^{\text{op}}$ denotes the opposite ring of $R$.

Throughout the rest of this talk, $R$ stands for a right Noetherian ring. We denote by $G_R$ the full subcategory of $\text{mod}_R$ consisting of modules $X \in \text{mod}_R$ with $\text{Ext}^i_R(X, R) = 0$ for all $i \geq 1$ and by $G^0_R$ the full subcategory of $G_R$ consisting of $X \in G_R$ with $\text{Hom}_R(X, R) = 0$. Then the generalized Nakayama conjecture is equivalent to that $G^0_R$ would not contain any simple module.

1. Preliminaries

Let $\{S_\lambda\}_{\lambda \in \Lambda}$ be a complete set of non-isomorphic simple modules in $\text{Mod}^{\text{op}}_R$ and $E_\lambda = E_{R^{\text{op}}}(S_\lambda)$ the injective envelope of $S_\lambda$ in $\text{Mod}^{\text{op}}_R$ for each $\lambda \in \Lambda$.

Lemma 1. For any $M \in \text{Mod}^{\text{op}}_R$ the following are equivalent.

1. $M = 0$.
2. $\text{Hom}_{R^{\text{op}}}(M, E_\lambda) = 0$ for all $\lambda \in \Lambda$.

Lemma 2. For any $X \in \text{mod}_R$ and any injective $E \in \text{Mod}^{\text{op}}_R$ we have

$X \otimes_R E \overset{\sim}{\rightarrow} \text{Hom}_{R^{\text{op}}}(\text{Hom}_R(X, R), E)$.

Lemma 3. For any $X \in \text{mod}_R$ the following are equivalent.

1. $\text{Hom}_R(X, R) = 0$.
2. $X \otimes_R E_\lambda = 0$ for all $\lambda \in \Lambda$.

Lemma 4. For any $X \in \text{mod}_R$ and any injective $E \in \text{Mod}^{\text{op}}_R$ we have

$\text{Tor}^R_i(X, E) \overset{\sim}{\rightarrow} \text{Hom}_{R^{\text{op}}}(\text{Ext}^R_i(X, R), E)$

for all $i \geq 0$.

The detailed version of this paper will be submitted for publication elsewhere.
Lemma 5. For any $X \in \text{mod-}R$ and any family of injectives $\{E_i\}_{i \in I}$ in $\text{Mod-}R^{\text{op}}$ we have

$$X \otimes_R \prod_{i \in I} E_i \cong \prod_{i \in I} X \otimes_R E_i.$$ 

2. Main results

Throughout the rest of this talk, $A$ is another right Noetherian ring and $V$ is an $(A, R)$-bimodule satisfying the following three conditions:

(a) $V \in \mathcal{G}_R$ in $\text{Mod-}R$.

(b) $V \in \text{Mod-}A^{\text{op}}$ is faithfully flat.

(c) $\text{inj dim} \, V \otimes_R E_\lambda < \infty$ in $\text{Mod-}A^{\text{op}}$ for all $\lambda \in \Lambda$.

Remark 6. If $\text{Hom}_R(V, R) \in \text{Mod-}A$ has finite projective dimension then the condition (c) is satisfied.

Lemma 7. We have $X \otimes_A V \in \mathcal{G}_R$ for all $X \in \mathcal{G}_A$.

Theorem 8. The following hold.

1. If $\mathcal{G}_R^0 = \{0\}$ then $\mathcal{G}_A^0 = \{0\}$.

2. If $\mathcal{G}_R$ consists only of torsionless modules then so does $\mathcal{G}_A$.

Remark 9. If $R$ is a left and right Noetherian ring and if $\mathcal{G}_R$ consists only of torsionless modules then every $X \in \mathcal{G}_R$ is Gorenstein projective (see e.g. [4]).

Corollary 10. Assume that for any maximal right ideal $m$ in $A$, setting $\mathfrak{A} = \{x \in R \mid Vx \subseteq mV\}$, $R/\mathfrak{A}$ is a semisimple ring. If the generalized Nakayama conjecture is true for $R$ then so is for $A$.

References


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