ON MODULES OF INFINITE REDUCED GRADE

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Abstract. Let R, A be right Noetherian rings and V an (A, R)-bimodule. Our aim is to provide a sufficient condition on V which enables A to inherit from R certain homological properties. Especially, we will show that if the generalized Nakayama conjecture is true for R then so is for A.

Introduction

In this talk we are mainly concerned with the generalized Nakayama conjecture which states that if R is a right Noetherian ring then every simple right R-module S with $\mathrm{Ext}^i_R(S, R) = 0$ for all $i \geq 1$ would be torsionless, i.e., $\mathrm{Hom}_R(S, R) \neq 0$ (see [1]).

For any ring R, we denote by Mod-R the category of right R-modules and by mod-R the full subcategory of Mod-R consisting of finitely presented modules, and left R-modules are considered as right R^{op}-modules, where R^{op} denotes the opposite ring of R.

Throughout the rest of this talk, R stands for a right Noetherian ring. We denote by $\mathcal{G}_R$ the full subcategory of mod-R consisting of modules $X \in$ mod-R with $\mathrm{Ext}^i_R(X, R) = 0$ for all $i \geq 1$ and by $\mathcal{G}^0_R$ the full subcategory of $\mathcal{G}_R$ consisting of $X \in \mathcal{G}_R$ with $\mathrm{Hom}_R(X, R) = 0$. Then the generalized Nakayama conjecture is equivalent to that $\mathcal{G}^0_R$ would not contain any simple module.

1. Preliminaries

Let $\{S_\lambda\}_{\lambda\in\Lambda}$ be a complete set of non-isomorphic simple modules in Mod-R^{op} and $E_\lambda = E_{R^{\mathrm{op}}}(S_\lambda)$ the injective envelope of S_λ in Mod-R^{op} for each $\lambda \in \Lambda$.

Lemma 1. *For any $M \in$ Mod-R^{op} the following are equivalent.*

(1) $M = 0$.

(2) $\mathrm{Hom}_{R^{\mathrm{op}}}(M, E_\lambda) = 0$ *for all* $\lambda \in \Lambda$.

Lemma 2. *For any $X \in$ mod-R and any injective $E \in$ Mod-R^{op} we have*

$$X \otimes_R E \xrightarrow{\sim} \mathrm{Hom}_{R^{\mathrm{op}}}(\mathrm{Hom}_R(X, R), E).$$

Lemma 3. *For any $X \in$ mod-R the following are equivalent.*

(1) $\mathrm{Hom}_R(X, R) = 0$.

(2) $X \otimes_R E_\lambda = 0$ *for all* $\lambda \in \Lambda$.

Lemma 4. *For any $X \in$ mod-R and any injective $E \in$ Mod-R^{op} we have*

$$\mathrm{Tor}^R_i(X, E) \xrightarrow{\sim} \mathrm{Hom}_{R^{\mathrm{op}}}(\mathrm{Ext}^i_R(X, R), E)$$

for all $i \geq 0$.

The detailed version of this paper will be submitted for publication elsewhere.

Lemma 5. *For any $X \in$ mod-R and any family of injectives $\{E_i\}_{i\in I}$ in* Mod-R^{op} *we have*

$$X \otimes_R \prod_{i\in I} E_i \xrightarrow{\sim} \prod_{i\in I} X \otimes_R E_i.$$

2. Main results

Throughout the rest of this talk, A is another right Noetherian ring and V is an (A, R)-bimodule satisfying the following three conditions:

(a) $V \in \mathcal{G}_R$ in Mod-R.
(b) $V \in$ Mod-A^{op} is faithfully flat.
(c) inj dim $V \otimes_R E_\lambda < \infty$ in Mod-A^{op} for all $\lambda \in \Lambda$.

Remark 6. If $\mathrm{Hom}_R(V, R) \in$ Mod-A has finite projective dimension then the condition (c) is satisfied.

Lemma 7. *We have $X \otimes_A V \in \mathcal{G}_R$ for all $X \in \mathcal{G}_A$.*

Theorem 8. *The following hold.*

(1) *If $\mathcal{G}^0_R = \{0\}$ then $\mathcal{G}^0_A = \{0\}$.*
(2) *If $\mathcal{G}_R$ consists only of torsionless modules then so does $\mathcal{G}_A$.*

Remark 9. If R is a left and right Noetherian ring and if $\mathcal{G}_R$ consists only of torsionless modules then every $X \in \mathcal{G}_R$ is Gorenstein projective (see e.g. [4]).

Corollary 10. *Assume that for any maximal right ideal $\mathfrak{m}$ in A, setting $\mathfrak{A} = \{x \in R \mid Vx \subset \mathfrak{m}V\}$, $R/\mathfrak{A}$ is a semisimple ring. If the generalized Nakayama conjecture is true for R then so is for A.*

References

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