EXTENSION GROUPS BETWEEN ATOMS AND CLASSIFICATION OF LOCALIZING SUBCATEGORIES

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Abstract. Gabriel showed that every localizing subcategory of the category of modules over a commutative noetherian ring is closed under injective envelopes. However, this does not necessarily hold for other locally noetherian Grothendieck categories, such as the category of graded modules over a noetherian graded ring or the category of right modules over a right noetherian ring. In this paper, for a locally noetherian Grothendieck category, we give a characterization of localizing subcategories closed under injective envelopes, based on a new concept: extension groups between atoms.

Key Words: Grothendieck category, Localizing subcategory, Atom spectrum, Extension group.

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1. Localizing subcategories

Throughout this paper, let $\mathcal{A}$ be a Grothendieck category.

Definition 1. A *localizing subcategory* of $\mathcal{A}$ is a full subcategory of $\mathcal{A}$ which is closed under subobjects, quotient objects, extensions, and arbitrary direct sums.

For a commutative noetherian ring R, Gabriel [1] gave a classification of localizing subcategories of the category $\operatorname{Mod} R$ of R-modules, and showed a property of them:

Theorem 2 ([1, Proposition VI.2.4]). *Let R be a commutative noetherian ring. Then there is a bijection*

$$\{\, \textit{localizing subcategories of } \operatorname{Mod} R \,\} \xrightarrow{\sim} \{\, \textit{specialization-closed subsets of } \operatorname{Spec} R \,\}$$

given by $\mathcal{X} \mapsto \bigcup_{M \in \mathcal{X}} \operatorname{Supp} M$. *The inverse map is given by* $\Phi \mapsto \{\, M \in \operatorname{Mod} R \mid \operatorname{Supp} M \subset \Phi \,\}$.

Theorem 3 ([1, Proposition V.5.10]). *Let R be a commutative noetherian ring. Then every localizing subcategory of $\operatorname{Mod} R$ is closed under injective envelopes.*

Theorem 2 has been generalized to locally noetherian Grothendieck category $\mathcal{A}$ in terms of the atom spectrum $\operatorname{ASpec} \mathcal{A}$ as Theorem 13.

On the other hand, Theorem 3 does not necessarily hold for a locally noetherian Grothendieck category. Even in the case of the module category Λ of a noncommutative artinian ring Λ, or in the case of the category $\operatorname{GrMod} A$ of $\mathbb{Z}$-graded modules over a

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commutative noetherian positively graded ring A, we can construct a localizing subcategory which is not closed under injective envelopes.

Example 4. Let $k[x]$ be the polynomial ring in a single variable x over a field k. Define a localizing subcategory $\mathcal{X}$ of $\operatorname{GrMod} k[x]$ by

$$\mathcal{X} = \{\, M \in \operatorname{GrMod} k[x] \mid M_i = 0 \text{ for all } i \in \mathbb{Z}_{<0} \,\}.$$

Then the $k[x]$-module $k[x]$ belongs to $\mathcal{X}$, but its injective envelope $E(k[x]) = k[x, x^{-1}]$ does not.

Example 5. Let Q be the quiver

$$1 \longrightarrow 2\,,$$

and consider the path algebra kQ over a field k. Define a localizing subcategory $\mathcal{Y}$ of $\operatorname{Mod} kQ$ by

$$\mathcal{Y} = \{\, M \in \operatorname{Mod} kQ \mid \operatorname{Hom}_{kQ}(M, E(S_1)) = 0 \,\},$$

where S_i is the simple right kQ-module corresponding to each vertex $i = 1, 2$. Then S_2 belongs to $\mathcal{Y}$, but its injective envelope $E(S_2)$ does not.

Note that the category $\operatorname{Mod} kQ$ is equivalent to the full subcategory

$$\{\, M \in \operatorname{GrMod} k[x] \mid M_i = 0 \text{ for all } i \notin \{1, 2\} \,\}$$

of $\operatorname{GrMod} k[x]$.

We will characterize localizing subcategories which are closed under injective envelopes in terms of atom spectrum.

2. Atom spectrum

In this section, we recall the definition of the atom spectrum of $\mathcal{A}$ and related notions.

The atom spectrum is defined by using monoform objects and an equivalence relation between them.

Definition 6.

(1) A nonzero object H in $\mathcal{A}$ is called *monoform* if for each nonzero subobject L of H, no nonzero subobject of H is isomorphic to a subobject of H/L.

(2) For monoform objects H_1 and H_2 in $\mathcal{A}$, we say that H_1 is *atom-equivalent* to H_2 if there exists a nonzero subobject of H_1 which is isomorphic to a subobject of H_2.

The following results are fundamental.

Proposition 7 ([3, section 2])**.**

(1) *Every nonzero subobject of a monoform object is monoform.*

(2) *Every monoform object H is uniform, that is, every nonzero subobject of H is essential.*

(3) *Every nonzero noetherian object has a monoform subobject. If $\mathcal{A}$ is a locally noetherian Grothendieck category, then every nonzero object has a monoform subobject.*

The atom equivalence is in fact an equivalence relation between the monoform objects in $\mathcal{A}$.

Definition 8. The *atom spectrum* $\operatorname{ASpec}\mathcal{A}$ of a Grothendieck category $\mathcal{A}$ is the quotient set of the set of monoform objects in $\mathcal{A}$ by the atom equivalence. Each element of $\operatorname{ASpec}\mathcal{A}$ is called an *atom* in $\mathcal{A}$. For each monoform object H in $\mathcal{A}$, the equivalence class of H is denoted by $\overline{H}$.

The notion of atoms was originally introduced by Storrer [6] for module categories and generalized to arbitrary abelian categories in [3].

The next result shows that the atom spectrum of a Grothendieck category is a generalization of the prime spectrum of a commutative ring.

Proposition 9 ([6, p. 631])**.** *Let R be a commutative ring. Then the map* $\operatorname{Spec} R \to \operatorname{ASpec}(\operatorname{Mod} R)$ *given by* $\mathfrak{p} \mapsto \overline{R/\mathfrak{p}}$ *is bijective.*

We can generalize the notion of support in terms of atoms.

Definition 10. Let M be an object in $\mathcal{A}$. Define the subset $\operatorname{ASupp} M$ of $\operatorname{ASpec}\mathcal{A}$ by

$$\operatorname{ASupp} M = \{\, \alpha \in \operatorname{ASpec}\mathcal{A} \mid \alpha = \overline{H} \text{ for some monoform subquotient } H \text{ of } M \,\}.$$

We call it the *atom support* of M.

Proposition 11. *Let R be a commutative ring, and let M be an R-module. Then the bijection* $\operatorname{Spec} R \to \operatorname{ASpec}(\operatorname{Mod} R)$ *in Proposition 9 induces a bijection* $\operatorname{Supp} M \to \operatorname{ASupp} M$.

For a locally noetherian Grothendieck category $\mathcal{A}$, the localizing subcategories of $\mathcal{A}$ can be classified by the localizing subsets of $\operatorname{ASpec}\mathcal{A}$.

Definition 12. A subset Φ of $\operatorname{ASpec}\mathcal{A}$ is called a *localizing subset* if there exists an object M in $\mathcal{A}$ such that $\Phi = \operatorname{ASupp} M$.

Theorem 13 ([2, Theorem 3.8], [5, Corollary 4.3], and [3, Theorem 5.5])**.** *Let $\mathcal{A}$ be a locally noetherian Grothendieck category. Then there is a bijection*

$$\{\, \textit{localizing subcategories of } \mathcal{A} \,\} \xrightarrow{\sim} \{\, \textit{localizing subsets of } \operatorname{ASpec}\mathcal{A} \,\}$$

given by $\mathcal{X} \mapsto \operatorname{ASupp}\mathcal{X} := \bigcup_{M\in\mathcal{X}} \operatorname{ASupp} M$. *The inverse map is given by* $\Phi \mapsto \operatorname{ASupp}^{-1}\Phi$, *where*

$$\operatorname{ASupp}^{-1}\Phi := \{\, M \in \mathcal{A} \mid \operatorname{ASupp} M \subset \Phi \,\}.$$

3. Extension groups between atoms

In this section, we determine which localizing subcategories are closed under injective envelopes, in terms of extension groups between atoms.

Definition 14. Let $\mathcal{A}$ be a Grothendieck category and $i \in \mathbb{Z}_{\geq 0}$.

(1) For each $\alpha = \overline{H} \in \operatorname{ASpec}\mathcal{A}$ and each object M in $\mathcal{A}$, we define the abelian group $\operatorname{Ext}^i_{\mathcal{A}}(\alpha, M)$ by

$$\operatorname{Ext}^i_{\mathcal{A}}(\alpha, M) = \varinjlim_{0\neq H'\subset H} \operatorname{Ext}^i_{\mathcal{A}}(H', M),$$

where H' runs over all nonzero subobjects of H. $\operatorname{Ext}^0_{\mathcal{A}}(\alpha, M)$ is also denoted by $\operatorname{Hom}_{\mathcal{A}}(\alpha, M)$.

(2) For each $\alpha = \overline{H}, \beta = \overline{L} \in \operatorname{ASpec} \mathcal{A}$, we define the abelian group $\operatorname{Ext}^i_{\mathcal{A}}(\alpha, \beta)$ by

$$\operatorname{Ext}^i_{\mathcal{A}}(\alpha, \beta) = \varprojlim_{0 \neq L' \subset L} \operatorname{Ext}^i_{\mathcal{A}}(\alpha, L'),$$

where L' runs over all nonzero subobjects of L. $\operatorname{Ext}^0_{\mathcal{A}}(\alpha, \beta)$ is also denoted by $\operatorname{Hom}_{\mathcal{A}}(\alpha, \beta)$.

(3) For each $\alpha \in \operatorname{ASpec} \mathcal{A}$, we define the *residue field* $k(\alpha)$ of α by $k(\alpha) = \operatorname{Hom}_{\mathcal{A}}(\alpha, \alpha)$.

Proposition 15. *Let R be a commutative noetherian ring, $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} R$, and let $\alpha, \beta \in \operatorname{ASpec}(\operatorname{Mod} R)$ be the atoms corresponding to $\mathfrak{p}, \mathfrak{q}$ by the bijection $\operatorname{Spec} R \to \operatorname{ASpec}(\operatorname{Mod} R)$ in Proposition 9, respectively.*

(1) *The functor $\operatorname{Ext}^i_R(\alpha, -)\colon \operatorname{Mod} R \to \operatorname{Mod} \mathbb{Z}$ is isomorphic to $\operatorname{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), (-)_\mathfrak{p})$, where $k(\mathfrak{p})$ is the residue field of $\mathfrak{p}$.*

(2) *$\operatorname{Ext}^i_R(\alpha, \beta) = 0$ whenever $\alpha \neq \beta$. $\operatorname{Ext}^i_R(\alpha, \alpha) \cong \operatorname{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), k(\mathfrak{p}))$ as an abelian group.*

(3) *$k(\alpha) \cong k(\mathfrak{p})$ as a field.*

For a Grothendieck category $\mathcal{A}$, the extension group $\operatorname{Ext}^i_{\mathcal{A}}(\alpha, M)$ between $\alpha \in \operatorname{ASpec} \mathcal{A}$ and $M \in \mathcal{A}$ has a structure of right $k(\alpha)$-module ([4, Theorem 4.7 (2)]) and $k(\alpha)$ is a skew field. We showed in [4] that its dimension is the corresponding Bass number.

Theorem 16 ([4, Theorem 5.3])**.** *Let $\mathcal{A}$ be a locally noetherian Grothendieck category, $\alpha \in \operatorname{ASpec} \mathcal{A}$, and let M be an object in $\mathcal{A}$. For each $i \in \mathbb{Z}_{\geq 0}$, we have the equation*

$$\mu_i(\alpha, M) = \dim_{k(\alpha)} \operatorname{Ext}^i_{\mathcal{A}}(\alpha, M),$$

where $\mu_i(\alpha, M)$ is the i-th Bass number of M with respect to α.

It turns out that the homomorphism space between distinct atoms is trivial.

Proposition 17. *Let $\mathcal{A}$ be a Grothendieck category and $\alpha, \beta \in \operatorname{ASpec} \mathcal{A}$. Then $\operatorname{Hom}_{\mathcal{A}}(\alpha, \beta) = 0$ whenever $\alpha \neq \beta$.*

Therefore the first nontrivial data between distinct atoms is the first extension group, and it leads us to the characterization of the localizing subcategories closed under injective envelopes.

Theorem 18. *Let $\mathcal{A}$ be a locally noetherian Grothendieck category. Then a localizing subcategory $\mathcal{X}$ of $\mathcal{A}$ is closed under injective envelopes if and only if $\Phi := \operatorname{ASupp} \mathcal{X}$ satisfies the following condition: if $\alpha \in \operatorname{ASpec} \mathcal{A}$ and $\beta \in \Phi$ satisfy $\operatorname{Ext}^1_{\mathcal{A}}(\alpha, \beta) \neq 0$, then $\alpha \in \Phi$.*

Example 19. In Example 4,

$$\operatorname{ASpec}(\operatorname{GrMod} k[x]) = \{\overline{k[x]}\} \cup \{\, \overline{k(i)} \mid i \in \mathbb{Z} \,\},$$

where $k(i)$ is the degree shift of k. The extension group $\operatorname{Ext}^1(\alpha, \beta)$ between atoms α, β is nonzero if and only if $(\alpha, \beta) = (\overline{k(i)}, \overline{k(i-1)})$ for some $i \in \mathbb{Z}$. The localizing subcategory $\mathcal{X}$ of $\operatorname{GrMod} k[x]$ corresponds to

$$\operatorname{ASupp} \mathcal{X} = \{\overline{k[x]}\} \cup \{\, \overline{k(i)} \mid i \in \mathbb{Z}_{\leq 0} \,\}.$$

Note that $\operatorname{Ext}^1(\overline{k(1)}, \overline{k}) = k$ while $\overline{k(1)} \notin \operatorname{ASupp} \mathcal{X}$ and $\overline{k} \in \operatorname{ASupp} \mathcal{X}$.

Example 20. In Example 5,

$$\mathrm{ASpec}(\mathrm{Mod}\, kQ) = \{\overline{S_1}, \overline{S_2}\},$$

and the extension group $\mathrm{Ext}^1(\alpha, \beta)$ between atoms α, β is nonzero if and only if $(\alpha, \beta) = (\overline{S_1}, \overline{S_2})$. The localizing subcategory $\mathcal{Y}$ of $\mathrm{Mod}\, kQ$ corresponds to

$$\mathrm{ASupp}\, \mathcal{Y} = \{\overline{S_2}\}.$$

Note that $\mathrm{Ext}^1(\overline{S_1}, \overline{S_2}) = k$ while $\overline{S_1} \notin \mathrm{ASupp}\, \mathcal{Y}$ and $\overline{S_2} \in \mathrm{ASupp}\, \mathcal{Y}$.

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