TILTING AND CLUSTER TILTING ASSOCIATED WITH REDUCED EXPRESSIONS IN COXETER GROUPS

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Abstract. Let Q be a finite acyclic quiver and Π be the preprojective algebra of Q. We study the stable category of a factor algebra of Π associated with an element w of the Coxeter group of Q. We see that the category has a silting object associated with each reduced expression of w. Under certain assumptions on a reduced expression, we see that the silting object is a tilting object.

1. Introduction

2-Calabi-Yau triangulated categories (2-CY, for short) and their cluster tilting objects have been studied extensively. For a given element w of the Coxeter group of a quiver, Buan-Iyama-Reiten-Scott constructed an Iwanaga-Gorenstein algebra $\Pi(w)$ and studied the stable category $\underline{\mathsf{Sub}}\,\Pi(w)$, where $\mathsf{Sub}\,\Pi(w)$ is the category of submodules of free $\Pi(w)$-modules [3]. They showed that $\underline{\mathsf{Sub}}\,\Pi(w)$ is a 2-CY triangulated category and constructed a cluster tilting object associated with each reduced expression of w.

There are other classes of 2-CY triangulated categories. Let A be a finite dimensional algebra which has a finite global dimension. Amiot introduced the generalized cluster category $\mathcal{C}(A)$ of A [1] and showed that if $\mathcal{C}(A)$ is Hom-finite, then $\mathcal{C}(A)$ is a 2-CY triangulated category and has cluster tilting objects.

In [2], Amiot-Reiten-Todorov studied the relationship between 2-CY triangulated categories $\underline{\mathsf{Sub}}\,\Pi(w)$ and $\mathcal{C}(A)$. For any element w of the Coxeter group and a reduced expression $\mathbf{w}$ of w, the authors constructed a finite dimensional algebra $A(\mathbf{w})$ and they showed that there exists a triangle equivalence $\mathsf{C}(A(\boldsymbol{w})) \simeq \underline{\mathsf{Sub}}\,\Pi(w)$.

The first aim of this paper is to introduce a graded analogue of an existence of cluster tilting objects of $\underline{\mathsf{Sub}}\,\Pi(w)$. Let w be an element of the Coxeter group of a quiver. We consider the stable category $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$ of a Frobenius category $\mathsf{Sub}^{\mathbb{Z}}\Pi(w)$, which is a graded analogue of $\underline{\mathsf{Sub}}\,\Pi(w)$. We show that the category $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$ has a silting object associated with each reduced expression of w. Under certain assumptions on a reduced expression of w, we show that the silting object is a tilting object. Finally, we compare the equivalence obtained by the tilting object and the equivalence of Amiot-Reiten-Todorov.

Notation. Through out this paper, let K be an algebraically closed field. All categories are K-linear categories. By a module, we mean a left module. For a ($\mathbb{Z}$-graded) ring A, we denote by $\mathsf{mod}A$ (respectively, $\mathsf{mod}^{\mathbb{Z}}A$) the category of finitely generated (respectively, $\mathbb{Z}$-graded) A-modules and by $\mathsf{proj}A$ (respectively, $\mathsf{proj}^{\mathbb{Z}}A$) the category of finitely generated (respectively, $\mathbb{Z}$-graded) projective A-modules. For $X \in \mathsf{mod}A$, we denote by $\mathsf{add}X$ the full subcategory of $\mathsf{mod}A$ whose objects are direct summands of finite direct sums of

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copies of X. For a triangulated category $\mathcal{T}$, we denote by [1] the shift functor of $\mathcal{T}$. For two arrows α, β of a quiver such that the target of α is the source of β, we denote by $\alpha\beta$ the composition of α and β.

2. Preliminaries

In this section, we give definitions used in this paper and recall some result of [3]. Throughout this section, let $Q = (Q_0, Q_1)$ be a finite acyclic quiver, where Q_0 is the set of vertices and Q_1 is the set of arrows. The *double quiver* $\overline{Q}$ is a quiver obtained from Q by adding an arrow $\alpha^* : v \to u$ for each arrow $\alpha : u \to v$ of Q. We define the preprojective algebra of Q and introduce a $\mathbb{Z}$-grading of it.

Definition 1. Let Q be a finite acyclic quiver. We define the *preprojective algebra* Π of Q by

$$\Pi := K\overline{Q}/\langle \sum_{\alpha\in Q_1} (\alpha\alpha^* - \alpha^*\alpha)\rangle.$$

We regard the path algebra $K\overline{Q}$ as a $\mathbb{Z}$-graded algebra by the following map deg: for each $\beta \in \overline{Q}_1$, let

$$\deg(\beta) = \begin{cases} 1 & \beta = \alpha^*,\ \alpha \in Q_1, \\ 0 & \beta = \alpha,\ \alpha \in Q_1. \end{cases}$$

Since the element $\sum_{\alpha\in Q_1}(\alpha\alpha^* - \alpha^*\alpha)$ of $K\overline{Q}$ is homogeneous of degree one, the grading of $K\overline{Q}$ naturally gives a grading on the preprojective algebra $\Pi = \bigoplus_{i\geq 0} \Pi_i$.

Let Q and Q' be finite acyclic quivers and Π and Π' be the preprojective algebras of Q and Q', respectively. It is easy to see that if the underlying graph of Q corresponds with that of Q', then Π is isomorphic to Π' as algebras, but not isomorphic to Π' as $\mathbb{Z}$-graded algebras, in general.

Next we define the Coxeter group of a quiver Q.

Definition 2. (1) The *Coxeter group* W_Q of Q is the group generated by the set $\{ s_u \mid u \in Q_0 \}$ with the following three relations: $s_u^2 = 1$, $s_u s_v = s_u s_v$ if there exist no arrows between u and v, and $s_u s_v s_u = s_v s_u s_v$ if there exists exactly one arrow between u and v.

(2) Let $w \in W_Q$ and $\mathbf{w} = s_{u_1} s_{u_2} \cdots s_{u_l}$ be an expression of w. $\mathbf{w}$ is said to be *reduced* if l is the smallest among expressions of w.

(3) For a reduced expression $\mathbf{w} = s_{u_1} s_{u_2} \ldots s_{u_l}$ of w, put $\mathsf{Supp}(w) := \{u_1, \ldots, u_l\}$. Note that, $\mathsf{Supp}(w)$ is independent of the choice of a reduced expression of w.

Next we define an algebra $\Pi(w)$, which is a factor algebra of Π. We first define a two-sided ideal of Π associated with an element w of W_Q. For each vertex $u \in Q_0$, we define a two-sided ideal I_u of Π by

$$I_u = \Pi(1 - e_u)\Pi,$$

where e_u is the idempotent of Π for u. Let $w \in W_Q$ and $\mathbf{w} = s_{u_1}s_{u_2}\dots s_{u_l}$ be a reduced expression of w. We define a two-sided ideal $I(w)$ of Π by

$$I(w) := I_{u_1}I_{u_2}\cdots I_{u_l}.$$

By [3, Theorem III. 1.9], $I(w)$ is independent of the choice of a reduced expression of w. Therefore we define an algebra $\Pi(w)$ for each $w \in W_Q$ by

$$\Pi(w) := \Pi/I(w).$$

It is easy to see that $\Pi(w)$ is a finite dimensional algebra for each $w \in W_Q$. Note that $\Pi(w)$ is a $\mathbb{Z}$-graded algebra, since each I_u and $I(w)$ are homogeneous ideals of Π.

We recall one property of $\Pi(w)$. A finite dimensional algebra A is said to be *Iwanaga-Gorenstein of dimension at most one* if $\mathsf{inj.dim}({}_AA) \leq 1$.

Proposition 3. [3] *For any element $w \in W_Q$, the algebra $\Pi(w)$ is Iwanaga-Gorenstein of dimension at most one.*

By Proposition 3, we have two Frobenius categories $\mathsf{Sub}\,\Pi(w)$ and $\mathsf{Sub}^{\mathbb{Z}}\Pi(w)$: the full subcategory of $\mathsf{mod}\,\Pi(w)$ (respectively, $\mathsf{mod}^{\mathbb{Z}}\Pi(w)$) consisting of submodules of free (respectively, graded free) $\Pi(w)$-modules of finite rank, where the projective-injective objects are $\mathsf{proj}\,\Pi(w)$ (respectively, $\mathsf{proj}^{\mathbb{Z}}\Pi(w)$). We denote by $\underline{\mathsf{Sub}}\,\Pi(w)$ and $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$ the stable categories of $\mathsf{Sub}\,\Pi(w)$ and $\mathsf{Sub}^{\mathbb{Z}}\Pi(w)$, which are triangulated. In the next section, we see that the triangulated category $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$ has a silting object.

3. Silting and tilting objects of $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$

In this section, we see that the triangulated category $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$ has a silting object for each reduced expression of $w \in W_Q$. First, we recall the definition of silting and tilting objects for triangulated categories. Let $\mathcal{T}$ be a triangulated category. For an object X of $\mathcal{T}$, we denote by $\mathsf{thick}\,X$ the smallest triangulated full subcategory of $\mathcal{T}$ containing X and closed under direct summands.

Definition 4. Let $\mathcal{T}$ be a triangulated category.

(1) An object X of $\mathcal{T}$ is called a *silting object* if $\mathsf{Hom}_{\mathcal{T}}(X, X[i]) = 0$ for any $0 < i$ and $\mathsf{thick}\,X = \mathcal{T}$.
(2) An object X of $\mathcal{T}$ is called a *tilting object* if X is a silting object of $\mathcal{T}$ and $\mathsf{Hom}_{\mathcal{T}}(X, X[i]) = 0$ for any $i < 0$.

We recall tilting theorem for triangulated categories. An additive category $\mathcal{C}$ is called *Krull-Schmidt* if each object of $\mathcal{C}$ is a finite direct sum of objects such that whose endomorphism algebras are local. For an algebra A, we denote by $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$ the bounded homotopy category of finitely generated projective A-modules.

Theorem 5. [4, (4.3)] *Let $\mathcal{T}$ be the stable category of a Frobenius category, and assume that $\mathcal{T}$ is Hom-finite and Krull-Schmidt. If there exists a tilting object X of $\mathcal{T}$, then we have a triangle equivalence $\mathcal{T} \simeq \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\mathsf{End}_{\mathcal{T}}(X))$.*

Note that our triangulated category $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$ satisfies the assumption of Theorem 5. Therefore by constructing a tilting object of $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$, we can realize this triangulated category as the derived category of a finite dimensional algebra. We first see that the category $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$ has a silting object.

Let $\mathbf{w} = s_{u_1}s_{u_2}\cdots s_{u_l}$ be a reduced expression of w, and put

$$M(\mathbf{w})^i = (\Pi/I(s_{u_1}s_{u_2}\cdots s_{u_i}))e_{u_i}, \quad M(\mathbf{w}) = \bigoplus_{i=1}^{l} M(\mathbf{w})^i.$$

We have the following theorem.

Theorem 6. [6] *Let $w \in W_Q$. For any reduced expression $\mathbf{w}$ of w, the object $M(\mathbf{w})$ is a silting object of $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$.*

Next we give a sufficient condition on $\mathbf{w}$ such that the silting object $M(\mathbf{w})$ is a tilting object. Let $\mathbf{w} = s_{u_1}s_{u_2}\cdots s_{u_l}$ be a reduced expression of w. For any $u \in \mathsf{Supp}(w)$, put

$$p_u := \max\{1 \le j \le l \mid u_j = u\}, \quad m_u := \min\{1 \le j \le l \mid u_j = u\}.$$

Definition 7. Let $\mathbf{w} = s_{u_1}s_{u_2}\cdots s_{u_l}$ be a reduced expression of w.

(1) We say that $\mathbf{w}$ is *c-ending* if for any $u, v \in \mathsf{Supp}(w)$, $p_u < p_v$ holds whenever there exists an arrow from u to v in Q.

(2) We say that $\mathbf{w}$ is *c-starting* if for any $u, v \in \mathsf{Supp}(w)$, $m_u < m_v$ holds whenever there exists an arrow from u to v in Q.

Example 8. Let $Q = \begin{tikzcd}[column sep=small] & 1 \arrow[dl] \arrow[dr] & \\ 2 \arrow[rr] & & 3 \end{tikzcd}$. An expression $\mathbf{w} = s_3s_2s_3s_1s_2s_3$ is a c-ending. Actually, $p_1 = 4$, $p_2 = 5$, and $p_3 = 6$. An expression $\mathbf{w}' = s_1s_2s_1s_3$ is c-starting. Actually, $p_1 = 1$, $p_2 = 2$ and $p_3 = 4$.

Then we can show the following theorem. For a finite dimensional algebra A, we denote by $\mathsf{D}^{\mathrm{b}}(A)$ the bounded derived category of the finitely generated A-modules.

Theorem 9. [6] *Let $\mathbf{w}$ be a reduced expression of w. Put $\mathcal{T} = \underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$ and $M = M(\mathbf{w})$. Assume that $\mathbf{w}$ is c-ending or c-starting. Then we have the following.*

(a) *M is a tilting object of $\mathcal{T}$.*

(b) *The global dimension of $\mathsf{End}_{\mathcal{T}}(M)$ is at most two.*

(c) *We have the following triangle equivalence*

$$\mathcal{T} \simeq \mathsf{D}^{\mathrm{b}}(\mathsf{End}_{\mathcal{T}}(M)).$$

Note that Theorem 9 (c) follows from Theorem 5 and Theorem 9 (b). There exists a more general condition on $\mathbf{w}$ such that $M(\mathbf{w})$ is a tilting object, see [6].

4. The relationship with the result of Amiot-Reiten-Todorov

In this section, we compare the equivalence obtained by the tilting object and the equivalence of Amiot-Reiten-Todorov. We first recall the cluster categories of finite dimensional algebras which are introduced by Amiot [1]. Let A be a finite dimensional algebra of global dimension at most two. We denote by $\mathbb{S} = - \otimes^{\mathbf{L}}_A DA$ a Serre functor on $\mathsf{D}^{\mathrm{b}}(A)$. Put $\mathbb{S}_2 = \mathbb{S} \circ [-2]$. A *cluster category* $\mathsf{C}(A)$ of A is the triangulated hull of the orbit category $\mathsf{D}^{\mathrm{b}}(A)/\mathbb{S}_2$ in the sense of Keller [5]. We have the composite of triangle functors

$$\pi_A : \mathsf{D}^{\mathrm{b}}(A) \to \mathsf{D}^{\mathrm{b}}(A)/\mathbb{S}_2 \to \mathsf{C}(A).$$

Let $\mathbf{w}$ be a reduced expression of $w \in W_Q$. Put $\mathcal{T} = \underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$ and $A = \mathsf{End}_{\mathcal{T}}(M(\mathbf{w}))$. We denote by e_i the idempotent of A associated with $M(\mathbf{w})^i$ for each $1 \leq i \leq l$. Let $e_F = \sum_{j \in F} e_j$, where $F = \{p_u \mid u \in \mathsf{Supp}(w)\}$. Put

$$\underline{A} = A/Ae_FA.$$

Amiot-Reiten-Todorov showed the following theorem.

Theorem 10. [2] *Let $w \in W_Q$ and $\mathbf{w}$ be a reduced expression of w. If $\mathbf{w}$ is c-ending, then we have the following.*

(a) *The global dimension of $\underline{A}$ is at most two.*
(b) *There exists a triangle equivalence $G : \mathsf{C}(\underline{A}) \to \underline{\mathsf{Sub}}\,\Pi(w)$.*

Then we have the following theorem.

Theorem 11. [6] *Let $w \in W_Q$ and $\mathbf{w}$ be a reduced expression of w. If $\mathbf{w}$ is c-ending, then $\mathsf{End}^{\mathbb{Z}}_{\mathcal{T}}(M(\mathbf{w})) = \underline{A}$ holds and we have the following commutative diagram up to isomorphism of functors*

$$\begin{tikzcd}
\mathsf{D}^{\mathrm{b}}(\underline{A}) \arrow[r, "\simeq"] \arrow[d, "\pi_{\underline{A}}"] & \underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w) \arrow[d, "\text{Forget}"] \\
\mathsf{C}(\underline{A}) \arrow[r, "G"] & \underline{\mathsf{Sub}}\,\Pi(w).
\end{tikzcd}$$

Finally, we give an example. For a graded module X and $i \in \mathbb{Z}$, we define the graded module $X(i)$ by $X(i)_j := X_{i+j}$.

Example 12. Let $Q = \begin{tikzcd}[column sep=small, row sep=small] & 1 \arrow[dl] \arrow[dr] & \\ 2 \arrow[rr] & & 3 \end{tikzcd}$. An expression $\mathbf{w} = s_3s_2s_3s_1s_2s_3$ is a c-ending. Let w be an element of W_Q which has a reduced expression $\mathbf{w}$. Put $\mathcal{T} = \underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$ and $M^i := M(\mathbf{w})^i$. It is easy to see that for any $u \in \mathsf{Supp}(w)$, $M^{p_u} \simeq \Pi(w)e_u$ holds. Therefore we have $M(\mathbf{w}) \simeq M^1 \oplus M^2 \oplus M^3$ in $\mathcal{T}$. By a direct calculation, we see that the endomorphism algebra $\mathsf{End}_{\mathcal{T}}(M(\mathbf{w}))$ is given by the following quiver with relations

$$\Delta = \bullet \xleftarrow{b} \bullet \xleftarrow{a} \bullet \qquad ab = 0.$$

This algebra is derived equivalent to the path algebra $K\Delta$. Therefore by Theorem 9, $\mathcal{T}$ is triangle equivalent to the derived category of the path algebra $K\Delta$. We can describe the Auslander-Reiten quiver of $\mathcal{T}$ as follows:

$$\begin{tikzcd}[column sep=small]
 & M^3[1](-1) \arrow[dr] & & M^3 \arrow[dr] & & M^2[1] \arrow[dr] & & M^2(1) \arrow[dr] & & M^2[-1](2) \arrow[dr] & \\
\cdots \arrow[ur] \arrow[dr] & & S_2 \arrow[ur] \arrow[dr] & & S_2[-1](1) \arrow[ur] \arrow[dr] & & S_2[1] \arrow[ur] \arrow[dr] & & S_2(1) \arrow[ur] \arrow[dr] & & \cdots \\
 & M^2 \arrow[ur] & & M^2[-1](1) \arrow[ur] & & M^1 \arrow[ur] & & M^3[1] \arrow[ur] & & M^3(1) \arrow[ur] &
\end{tikzcd}$$

where S_2 is a simple $\Pi(w)$-module associated with a vertex $2 \in Q_0$.

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