CLASSIFYING DENSE SUBCATEGORIES OF EXACT CATEGORIES VIA GROTHENDIECK GROUPS

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Abstract. Classification problems of subcategories have been deeply considered so far. In this article, we discuss classifying dense resolving subcategories of exact categories via their Grothendieck groups. This study is motivated by the classification of dense triangulated subcategories of triangulated categories due to Thomason.

1. Introduction

Classification of subcategories means for a category $\mathcal{C}$, finding a bijection

$$\{\cdots \text{ subcategories of } \mathcal{C}\} \rightleftarrows S,$$

where S is a set which is easier to understand. Classification of subcategories is an important approach to understand the category $\mathcal{C}$ and has been studied in various areas of mathematics, for example: stable homotopy theory, commutative/nonncommutative ring theory, algebraic geometry, modular representation theory.

Let $\mathcal{C}$ be an additive category, $\mathcal{X}$ an additive full subcategory. We say that $\mathcal{X}$ is an *additively closed* subcategory of $\mathcal{C}$ if it is closed under taking direct summands, and that $\mathcal{X}$ is a *dense* subcategory of $\mathcal{C}$ if any object of $\mathcal{C}$ is a direct summand of some object of $\mathcal{X}$. We can easily check that $\mathcal{X}$ is additively closed if and only if $\mathcal{X} = \mathsf{add}\,\mathcal{X}$ and that $\mathcal{X}$ is dense if and only if $\mathcal{C} = \mathsf{add}\,\mathcal{X}$. Here, we denote by $\mathsf{add}\,\mathcal{X}$ the additive closure of $\mathcal{X}$, namely, $\mathsf{add}\,\mathcal{X}$ is a subcategory of $\mathcal{C}$ consisting of objects which are direct summands of finite direct sums of objects of $\mathcal{X}$. For this reason, to classify additive subcategories, it suffice to classify additively closed ones and dense ones. Following theorem is an example of classification of dense subcategories which is due to Thomason [5].

Theorem 1 (Thomason)**.** *Let $\mathcal{T}$ be an essentially small triangulated category. Then there is a one to one correspondence*

$$\{\textit{dense triangulated subcategories of } \mathcal{T}\} \underset{g}{\overset{f}{\rightleftarrows}} \{\textit{subgroups of } K_0(\mathcal{T})\}\ .$$

Motivated by this theorem, we discuss classifying dense resolving subcategories of exact categories.

2. classification of dense resolving subcategories

In this section, we give our main result and several corollaries.

The detailed version [4] of this article will be submitted for publication elsewhere.

Let us begin with fixing our conventions. Throughout this article, $\mathcal{E}$ denotes an exact category. We always assume that all categories are essentially small, and that all subcategories are full and additive. For a left noetherian ring A, denote by $\mathsf{mod}\, A$ the abelian category of finitely generated left A-modules, and by $\mathsf{proj}\, A$ its full subcategory consisting of finitely generated projective left A-modules.

Next we recall the definition of the Grothendieck groups of an exact category and a triangulated category.

Definition 2. (1) We define the *Grothendieck group* of an exact category $\mathcal{E}$ by

$$K_0(\mathcal{E}) := \frac{\bigoplus_{X \in \mathcal{E}/\cong} \mathbb{Z} \cdot X}{\langle X - Y + Z \mid X \rightarrowtail Y \twoheadrightarrow Z \text{ is a short exact sequence of } \mathcal{E} \rangle}.$$

(2) We define the *Grothendieck group* of a triangulated category $\mathcal{T}$ by

$$K_0(\mathcal{T}) := \frac{\bigoplus_{X \in \mathcal{T}/\cong} \mathbb{Z} \cdot X}{\langle X - Y + Z \mid X \to Y \to Z \to \Sigma X \text{ is an exact triangle of } \mathcal{T} \rangle}.$$

For an object X, denote by $[X]$ the corresponding element in the Grothendieck group.

Let us recall the definition of generators of an exact category.

Definition 3. Let $\mathcal{G}$ be a class of objects of $\mathcal{E}$. We call $\mathcal{G}$ a *generator* of $\mathcal{E}$ if for any object $A \in \mathcal{E}$, there is a short exact sequence

$$A' \rightarrowtail G \twoheadrightarrow A$$

in $\mathcal{E}$ with $G \in \mathcal{G}$.

Example 4. (1) Clearly, $\mathcal{E}$ is a generator of $\mathcal{E}$

(2) Denote by $\mathsf{proj}\, \mathcal{E}$ the subcategory of $\mathcal{E}$ consisting of projective objects. Then by definition, $\mathsf{proj}\, \mathcal{E}$ is a generator of $\mathcal{E}$ if and only if $\mathcal{E}$ has enough projective objects.

Finally, we introduce the notion of $\mathcal{G}$-resolving subcategories of an exact category.

Definition 5. Let $\mathcal{X}$ be a subcategory of $\mathcal{E}$ and $\mathcal{G}$ a generator of $\mathcal{E}$. We say that $\mathcal{X}$ is a $\mathcal{G}$*-resolving* subcategory of $\mathcal{E}$ if:

(1) $\mathcal{X}$ is closed under kernels of admissible epimorphisms,
(2) $\mathcal{X}$ is closed under extensions, and
(3) $\mathcal{X}$ contains $\mathcal{G}$.

The following theorem is our main result in this article.

Theorem 6. *Let $\mathcal{E}$ be an essentially small exact category with a generator $\mathcal{G}$. Then there are bijections*

$$\left\{ \begin{matrix} \text{dense } \mathcal{G}\text{-resolving subcategories} \\ \text{of } \mathcal{E} \end{matrix} \right\} \underset{g}{\overset{f}{\rightleftarrows}} \left\{ \begin{matrix} \text{subgroups of } K_0(\mathcal{E}) \\ \text{containing } \langle [G] \mid G \in \mathcal{G} \rangle \end{matrix} \right\},$$

where f and g are given by $f(\mathcal{X}) := \langle [X] \mid X \in \mathcal{X} \rangle$ and $g(H) := \{X \in \mathcal{E} \mid [X] \in H\}$, respectively.

Sketch of proof. Let H be a subgroup of $K_0(\mathcal{E})$ containing $\langle [G] \mid G \in \mathcal{G} \rangle$. Clearly, $g(H)$ contains $\mathcal{G}$. For a short exact sequence $A \rightarrowtail B \twoheadrightarrow C$, one has $[A] - [B] + [C] = 0 \in H$. This shows that $g(H)$ is a $\mathcal{G}$-resolving subcategory of $\mathcal{E}$. Moreover, for $A \in \mathcal{G}$, take a short exact sequence $A' \rightarrowtail G \twoheadrightarrow A$ in $\mathcal{E}$ with $G \in \mathcal{G}$. Then we have $[A \oplus A'] = [A] + [A'] = [G] \in \langle [G] \mid G \in \mathcal{G} \rangle \subseteq H$. Thus, A is a direct summand of $A \oplus A'$ which belongs to $g(H)$. Consequently, $g(H)$ is a dense $\mathcal{G}$-resolving subcategory.

We give an outline of the proof of $gf = id$. Fix a dense $\mathcal{G}$-resolving subcategory $\mathcal{X}$ of $\mathcal{E}$. Define an equivalence relation $\sim$ on $\mathcal{E}/\cong$ as follows:

$$A \sim A' :\Leftrightarrow \exists X, X' \in \mathcal{X} \text{ such that } A \oplus X \cong A' \oplus X'.$$

Denote by $\langle \mathcal{E} \rangle_{\mathcal{X}}$ the quotient set of $\mathcal{E}/\cong$ by $\sim$. For $A \in \mathcal{E}$, $\langle A \rangle$ shall mean the class of A in $\langle \mathcal{E} \rangle_{\mathcal{X}}$. Then we can show:

(1) $A \sim 0 \Leftrightarrow A \in \mathcal{X}$.
(2) $\langle \mathcal{E} \rangle_{\mathcal{X}}$ is an abelian group by $\langle A \rangle + \langle B \rangle := \langle A \oplus B \rangle$.
(3) $\varphi : K_0(\mathcal{E}) \to \langle \mathcal{E} \rangle_{\mathcal{X}}, [A] \mapsto \langle A \rangle$ is a well-defined homomorphism of abelian groups.
(4) $\mathsf{Ker}\, \varphi = f(\mathcal{X})$.

Here, we use the assumption '$\mathcal{X}$ is a dense $\mathcal{G}$-resolving' to show these statement.

Consequently, we obtain:

$$A \in \mathcal{X} \Leftrightarrow [A] \in f(\mathcal{X}) \Leftrightarrow A \in gf(\mathcal{X}).$$

First equivalence is due to (1), (4) and the second one is by definition. $\square$

For the lest of this article, we give some corollaries and applications of our main theorem.

To begin with, we state the following lemma which is well-known for the case where $\mathcal{E}$ is abelian.

Lemma 7. *Let $\mathcal{E}$ be a weakly idempotent complete exact category. Then the natural functor $\mathcal{E} \to \mathsf{D}^{\mathsf{b}}(\mathcal{E})$ induces an isomorphism*

$$K_0(\mathcal{E}) \cong K_0(\mathsf{D}^{\mathsf{b}}(\mathcal{E}))$$

Combination of Theorem 1 and Theorem 6 shows the following corollary.

Corollary 8. *Assume that $\mathcal{E}$ is a weakly idempotent complete exact category with a generator $\mathcal{G}$. Then there are one-to-one correspondences among the following sets:*

(1) $\{$*dense $\mathcal{G}$-resolving subcategories of $\mathcal{E}$*$\}$.
(2) $\{$*dense triangulated subcategories of $\mathcal{T}$ containing $\mathcal{G}$*$\}$.
(3) $\{$*subgroups of $K_0(\mathcal{E})$ containing $\langle [G] \mid G \in \mathcal{G} \rangle$*$\}$.

Remark 9. Let $\mathcal{X}$ be a dense subcategory of $\mathcal{E}$ which is closed under kernels of admissible epimorphisms and extensions. Then we can easily check that it is automatically closed under cokernels of admissible monomorphisms. Thus, if we take $\mathcal{G} = \mathsf{proj}\, \mathcal{E}$, the above corollary can be considered as the dense version of the following theorem due to Krause and Stevenson [4]:

Theorem 10 (Krause-Stevenson)**.** *Let $\mathcal{E}$ be an exact category with enough projective objects. Then there is one-to-one correspondence between:*

(1) $\{$*thick subcategories of $\mathcal{E}$ containing* $\mathsf{proj}\, \mathcal{E}$$\}$*, and*

(2) $\{$*thick subcategories of* $\mathsf{D}^{\mathsf{b}}(\mathcal{E})$ *containing* $\mathsf{proj}\,\mathcal{E}\}$.

Next, we apply our main theorem for module categories of Iwanaga-Gorenstein rings.

Let S be an *Iwanaga-Gorenstein ring* (i.e. S is noetherian on both sides and S is of finite injective dimension as a left S-module and a right S-module). Let us give several remarks about Iwanaga-Gorenstein rings (cf. [2, 6]).

Remark 11. (1) We say that a finitely generated left S-module X is *maximal Cohen-Macaulay* if $\mathsf{Ext}^i_S(X, S) = 0$ for all integers $i > 0$. $\mathsf{CM}(S)$ denotes the subcategory of $\mathsf{mod}\,S$ consisting of maximal Cohen-Macaulay S-modules. Then it is a Frobenius category, and hence, its stable category $\underline{\mathsf{CM}}(S)$ is triangulated.

(2) Natural inclusions $\mathsf{CM}(S) \hookrightarrow \mathsf{mod}\,S \hookrightarrow \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,S)$ induce isomorphisms

$$K_0(\mathsf{CM}(S)) \cong \mathsf{K}_0(\mathsf{mod}\,S) \cong K_0(\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,S)).$$

(3) Composition of the natural inclusion $\mathsf{CM}(S) \hookrightarrow \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,S)$ and the quotient functor $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,S) \to \mathsf{D}_{\mathsf{sg}}(S) := \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,S)/\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,S)$ induces a triangle equivalence

$$\underline{\mathsf{CM}}(S) \cong \mathsf{D}_{\mathsf{sg}}(S).$$

From these remarks, we obtain the following corollary.

Corollary 12. *Let S be an Iwanaga-Gorenstein ring. Then there are one-to-one correspondences among the following sets:*

(1) $\{$*dense resolving subcategories of* $\mathsf{CM}(S)\}$,
(2) $\{$*dense resolving subcategories of* $\mathsf{mod}\,S\}$,
(3) $\{$*dense triangulated subcategories of* $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,S)$ *containing* $\mathsf{proj}\,S\}$,
(4) $\{$*dense triangulated subcategories of* $\mathsf{D}_{\mathsf{sg}}(S)\}$,
(5) $\{$*dense triangulated subcategories of* $\underline{\mathsf{CM}}(S)\}$, *and*
(6) $\{$*subgroups of* $K_0(\mathsf{mod}\,S)$ *containing* $\langle [P] \mid P \in \mathsf{proj}\,S\rangle\}$

3. Finiteness of the number of dense resolving subcategories

In this section, we discuss when mod A has only finitely many dense resolving subcategories.

First consider the case of finite dimensional algebra. Let A be a basic finite dimensional algebra over a field k with a complete set $\{e_1, \ldots, e_n\}$ of primitive orthogonal idempotents. Denote by $C_A := (\mathsf{dim}_k\, e_iAe_j)_{i,j=1,\ldots,n}$ the *Cartan matrix* of A. Then simple A-modules $\{S_i := Ae_i/\,\mathsf{rad}(\mathrm{Ae_i})\}_{i=1}^n$ forms a free basis of the Grothendieck group $K_0(\mathsf{mod}\,A)$, and hence there is an isomorphism of abelian groups:

$$K_0(\mathsf{mod}\,A) \cong \mathbb{Z}^{\oplus n}.$$

Furthermore, this isomorphism induces an isomorphism:

$$K_0(\mathsf{mod}\,A)/\langle [P] \mid P \in \mathsf{proj}\,A\rangle \cong \mathsf{Coker}(\mathbb{Z}^{\oplus n} \xrightarrow{C_A} \mathbb{Z}^{\oplus n})$$

for details, see [1]. From this argument, we have the following corollary.

Corollary 13. *Let A be a basic finite dimensional k-algebra. The following are equivalent:*

(1) *There are only finitely many dense resolving subcategories of* $\mathsf{mod}\,A$.
(2) $\mathsf{det}\,C_A \neq 0$.

Next consider the case of simple singularities. Let k be an algebraically closed field of characteristic 0. We say that a two dimensional commutative noetherian local ring $R := k[[x, y, z]]/(f)$ has a *simple singularity* if f is one of the following form:

$$\begin{array}{ll} (\mathsf{A}_n) & x^2 + y^{n+1} + z^2 \ (n \geq 1), \\ (\mathsf{D}_n) & x^2y + y^{n-1} + z^2 \ (n \geq 4), \\ (\mathsf{E}_6) & x^3 + y^4 + z^2, \\ (\mathsf{E}_7) & x^3 + xy^3 + z^2, \\ (\mathsf{E}_8) & x^3 + y^5 + z^2. \end{array}$$

In this case the Grothendieck group $K_0(\operatorname{\mathsf{mod}} R)$ is given as follows (see [6, Proposition 13.10]):

	$K_0(\operatorname{\mathsf{mod}} R)$	$\#\{$ dense resolv. subcat. of $\operatorname{\mathsf{mod}} R\}$
(A_n)	$\mathbb{Z} \oplus \mathbb{Z}/(n+1)\mathbb{Z}$	the number of divisors of $n+1$
(D_n) ($n =$ even)	$\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	5
(D_n) ($n =$ odd)	$\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$	3
(E_6)	$\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$	2
(E_7)	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	2
(E_8)	$\mathbb{Z}$	1

Here, $\mathbb{Z}$ appearing in $K_0(\operatorname{\mathsf{mod}} R)$ is generated by $[R]$. Thus, our main theorem implies that there are only finitely many dense resolving subcategories of $\operatorname{\mathsf{mod}} R$ for a two dimensional simple singularity R. Actually, the converse also holds true under some mild assumptions.

Corollary 14. *Let R be a 2-dimensional complete Gorenstein normal local ring with algebraically closed residue field k of characteristic 0. The following are equivalent:*

1. *There are only finitely many dense resolving subcategories of* $\operatorname{\mathsf{mod}} R$.
2. *There are only finitely many dense resolving subcategories of* $\operatorname{\mathsf{mod}} R$ *containing* k.
3. *R has a simple singularity.*

Finally, I give the complete set of dense resolving subcategories of $\operatorname{\mathsf{mod}} R$ for a simple singularity of type (A_1).

Example 15. Let $R := k[[x, y, z]]/(x^2 + y^2 + z^2)$ be a simple singularity of type (A_1). Then indecomposable maximal Cohen-Macaulay R-modules are R and $I := (x+\sqrt{-1}y, z)$. Then dense resolving subcategories of $\operatorname{\mathsf{mod}} R$ are:

- $\operatorname{\mathsf{mod}} R$, and
- $\{M \in \operatorname{\mathsf{mod}} R \mid \Omega^2 M \cong R^{\oplus m} \oplus I^{\oplus 2n}$ for $\exists m, n \in \mathbb{Z}_{\geq 0}\}$.

Here we denote by $\Omega^2 M$ the second syzygy module of M.

Proof. From Corollary 12, there is a one-to-one correspondence between the set of dense resolving subcategories of $\operatorname{\mathsf{mod}} R$ and the set of dense resolving subcategories of $\mathsf{CM}(R)$. This correspondence assigns a dense resolving subcategory $\mathcal{X}$ of $\mathsf{CM}(R)$ to a dense resolving subcategory $\{M \in \operatorname{\mathsf{mod}} R \mid \Omega^2 M \in \mathcal{X}\}$ of $\operatorname{\mathsf{mod}} R$. Therefore, it is enough to check that the dense resolving subcategories of $\mathsf{CM}(R)$ are $\mathsf{CM}(R)$ and $\{M \in \operatorname{\mathsf{mod}} R \mid M \cong R^{\oplus m} \oplus I^{\oplus 2n}$ for $\exists m, n \in \mathbb{Z}_{\geq 0}\}$.

To show this, we use the following lemma:

Lemma 16. *Consider a short exact sequence* $0 \to I^{\oplus m} \to W \to I^{\oplus n} \to 0$. *Then* $W \cong R^{\oplus 2i} \oplus I^{m+n-2i}$ *for some* $0 \leq i \leq \min\{m, n\}$.

Let $\mathcal{X} := \{M \in \mathsf{CM}(R) \mid M \cong R^{\oplus m} \oplus I^{\oplus 2n}$ for $\exists m, n \in \mathbb{Z}_{\geq 0}\}$. Then $\mathcal{X}$ is a dense subcategory of $\mathsf{CM}(R)$ because every maximal Cohen-Macaulay R-module is a finite direct sum of R and I. Moreover, this lemma shows that $\mathcal{X}$ is closed under extensions.

Consider a short exact sequence

$$0 \to R^{\oplus m_1} \oplus I^{\oplus n_1} \to R^{\oplus m_2} \oplus I^{\oplus 2n_2} \to R^{\oplus m_3} \oplus I^{\oplus 2n_3} \to 0.$$

This sequence is isomorphic to

$$(0 \to I^{\oplus n_1} \to R^{\oplus r} \oplus I^{\oplus 2n_2} \to I^{\oplus 2n_3} \to 0) \oplus (0 \to R^{\oplus m_1} \to R^{\oplus m_1+m_3} \to R^{\oplus m_3} \to 0)$$

since $\mathsf{Ext}^1_R(R, I) = \mathsf{Ext}^1_R(I, R) = 0$. Then, by Lemma 16, $R^{\oplus r} \oplus I^{\oplus 2n_2}$ must be isomorphic to

$$R^{\oplus 2i} \oplus I^{\oplus n_1+2n_3-2i}$$

for some integer i. Therefore, $2n_2 = n_1 + 2n_3 - 2i$ as $\mathsf{CM}(R)$ is a Krull-Schmidt category, see [6, Proposition 1.18]. Thus, $R^{\oplus m_1} \oplus I^{\oplus n_1} = R^{\oplus m_1} \oplus I^{\oplus 2(n_2-n_3+i)}$ belongs to $\mathcal{X}$. Consequently, $\mathcal{X}$ is a dense resolving subcategory of $\mathsf{CM}(R)$ which is not equal to $\mathsf{CM}(R)$.

Since R is a two dimensional simple singularity of type (A_1), $\operatorname{\mathsf{mod}} R$ has only two dense resolving subcategories; $K_0(\operatorname{\mathsf{mod}} R)/\langle [R] \rangle \cong \mathbb{Z}/2\mathbb{Z}$ has only two subgroups. As a consequence, dense resolving subcategories of $\mathsf{CM}(R)$ are $\mathsf{CM}(R)$ and $\mathcal{X}$. $\square$

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