RINGEL DUALITY AND RECOLLEMENTS

HIROYUKI MINAMOTO

Abstract. It was shown by Cline-Parshall-Scott that the derived category of a quasi-hereditary algebra is obtained from those of the residue fields by a sequence of recollements. We show that the sequence of recollements of the Ringel dual of a quasi-hereditary algebra is obtained from that of original quasi-hereditary algebra by a categorical operation. This observation gives a look of the results of Krause that twice of the Ringel duality is the Serre duality. Our observation enable us to generalize a notion of Ringel duality for finite dimensional algebra equipped with an appropriate sequence of recollements. We end this note by computing an example of generalized Ringel duality.

1. Introduction

It has been known by Cline-Parshall-Scott [3] that a quasi-hereditary algebra Λ is obtained by gluing its residue fields $\Gamma_1, \dots, \Gamma_n$. More precisely, there are a sequence of recollements

$$
\begin{aligned}
&\mathcal{D}^{\mathrm{b}}(\Gamma_1) \equiv \mathcal{D}^{\mathrm{b}}(\Lambda_2) \equiv \mathcal{D}^{\mathrm{b}}(\Gamma_2),\\
&\mathcal{D}^{\mathrm{b}}(\Lambda_2) \equiv \mathcal{D}^{\mathrm{b}}(\Lambda_3) \equiv \mathcal{D}^{\mathrm{b}}(\Gamma_3),\\
&\mathcal{D}^{\mathrm{b}}(\Lambda_3) \equiv \mathcal{D}^{\mathrm{b}}(\Lambda_4) \equiv \mathcal{D}^{\mathrm{b}}(\Gamma_4),\\
&\qquad\vdots \qquad\qquad \vdots \qquad\qquad \vdots\\
&\mathcal{D}^{\mathrm{b}}(\Lambda_{n-1}) \equiv \mathcal{D}^{\mathrm{b}}(\Lambda) \equiv \mathcal{D}^{\mathrm{b}}(\Gamma_n).
\end{aligned} \tag{1.1}
$$

Recently, Krause [7] determined the condition for sequence of recollements abelian categories of residue fields which ensure that Λ is quasi-hereditary.

Let Λ be a quasi-hereditary algebra. Since its Ringel dual $\mathsf{R}(\Lambda)$ is a quasi-hereditary algebra with the reverse order on the idempotents $e_1, e_2, \dots, e_n$, there is a sequence of recollements

$$
\begin{aligned}
&\mathcal{D}^{\mathrm{b}}(\Gamma_n) \equiv \mathcal{D}^{\mathrm{b}}(\Lambda'_2) \equiv \mathcal{D}^{\mathrm{b}}(\Gamma_{n-1}),\\
&\mathcal{D}^{\mathrm{b}}(\Lambda'_2) \equiv \mathcal{D}^{\mathrm{b}}(\Lambda'_3) \equiv \mathcal{D}^{\mathrm{b}}(\Gamma_{n-2}),\\
&\mathcal{D}^{\mathrm{b}}(\Lambda'_3) \equiv \mathcal{D}^{\mathrm{b}}(\Lambda'_4) \equiv \mathcal{D}^{\mathrm{b}}(\Gamma_{n-3}),\\
&\qquad\vdots \qquad\qquad \vdots \qquad\qquad \vdots\\
&\mathcal{D}^{\mathrm{b}}(\Lambda'_{n-1}) \equiv \mathcal{D}^{\mathrm{b}}(\mathsf{R}(\Lambda)) \equiv \mathcal{D}^{\mathrm{b}}(\Gamma_1).
\end{aligned}
$$

The detailed version of this paper will be submitted for publication elsewhere.

In this note, we show that we can get this sequence from the sequence (1.1) by categorical operation. In case of the number n of the idempotents is 2, (so the sequence consists of single recollement) this operation is nothing but the reflection due to Jorgensen [4] .

This observation gives a look of the results of Krause [6] that twice of the Ringel duality is the Serre duality [1]. (Corollary 9, Theorem 12.)

Our observation enable us to generalize a notion of Ringel duality for finite dimensional algebra equipped with an appropriate sequence of recollements. We end this note by computing an example of generalized Ringel duality.

2. Ringel dual of a bi-admissible filtration

In this section $\mathcal{T}$ denotes a triangulated category. Recall that a full sub triangulated category $\mathcal{S} \subset \mathcal{T}$ is called *bi-admissible* if the inclusion $\mathcal{S} \hookrightarrow \mathcal{T}$ has left and right adjoint functors.

Definition 1. A sequence $\mathcal{F} : 0 = \mathcal{F}_0 \hookrightarrow \mathcal{F}_1 \overset{I_1}{\hookrightarrow} \mathcal{F}_2 \overset{I_2}{\hookrightarrow} \cdots \overset{I_{n-1}}{\hookrightarrow} \mathcal{F}_n = \mathcal{T}$ of full sub triangulated categories is called a *bi-admissible filtration* if $\mathcal{F}_a$ is a bi-admissible subcategory of $\mathcal{F}_{a+1}$ for $a = 1, \dots, n-1$.

Definition 2. A bi-admissible echelon $\mathcal{E}$ with the top $\mathcal{T}$ of length n is a diagram which has $\frac{n(n+1)}{2}$ vertexes on which there are triangulated categories $(\mathcal{E}_{ab})_{1 \le b \le a \le n}$ with $\mathcal{E}_{1n} = \mathcal{T}$, and arrows are bi-admissible quotient functors $J^{\mathcal{E}}_{ab} : \mathcal{E}_{ab} \to \mathcal{E}_{a+1,b}$ and bi-admissible embedding functors $I^{\mathcal{E}}_{ab} : \mathcal{E}_{ab} \to \mathcal{E}_{a,b+1}$ which satisfy obvious commutativity, such that the diagrams are exact sequences of triangulated categories for all possible combinations of a, b, c

$$\mathcal{E}_{ab} \xrightarrow{I_{a,b+c-1} \cdots I_{a,b+1} I_{ab}} \mathcal{E}_{a,b+c} \xrightarrow{J_{a+b-1,b+c} \cdots J_{a+1,b+c} J_{a,b}} \mathcal{E}_{a+b,b+c}.$$

We give a picture of a bi-admissible echelon of length 3. We also give a picture of $\mathsf{E}(\mathcal{F})$ defined below.

$$\begin{tikzcd} \mathcal{E} : \mathcal{E}_{11} \arrow[r, "I_{11}"] & \mathcal{E}_{12} \arrow[r, "I_{12}"] \arrow[d, "J_{12}"'] & \mathcal{E}_{13} = \mathcal{T} \arrow[d, "J_{13}"] \\ & \mathcal{E}_{22} \arrow[r, "I_{22}"'] & \mathcal{E}_{23} \arrow[d, "J_{23}"] \\ & & \mathcal{E}_{33} \end{tikzcd} , \quad \begin{tikzcd} \mathsf{E}(\mathcal{F}) : \mathcal{F}_1 \arrow[r] & \mathcal{F}_2 \arrow[r] \arrow[d] & \mathcal{F}_3 = \mathcal{T} \arrow[d] \\ & \mathcal{F}_2/\mathcal{F}_1 \arrow[r] & \mathcal{F}_3/\mathcal{F}_1 \arrow[d] \\ & & \mathcal{F}_3/\mathcal{F}_2 \end{tikzcd}$$

From a bi-admissible filtration $\mathcal{F}$ of $\mathcal{T}$, we construct a bi-admissible echelon $\mathsf{E}(\mathcal{F})$ in the following way: at the (a, b)-th vertex we put the quotient categories $\mathcal{F}_b/\mathcal{F}_{a-1}$ and the functors I_{ab}, J_{ab} are set to be induced functors. This diagram become a bi-admissible echelon.

From a bi-admissible echelon $\mathcal{E}$ of the top $\mathcal{T}$, we construct a bi-admissible filtration $\mathsf{F}(\mathcal{E})$ of $\mathcal{T}$ by taking the top row of $\mathcal{E}$. Namely, we put $\mathcal{F}_b := \mathcal{E}_{1b}$ and $I_b := I_{1b}$.

The following Lemma follows from standard but tedious argument.

[1]For a special class of quasi-hereditary algebras, this was conjectured by Kapranov and proved by Beilinson-Bezrukavnikov-Mirković [2].

Lemma 3. *The above correspondences are the inverse to each other up to equivalences.*

$$\mathsf{FE}(\mathcal{F}) = \mathcal{F}, \mathsf{EF}(\mathcal{E}) = \mathcal{E}.$$

Let $\mathcal{E}$ be a bi-admissible echelon of length n of the top $\mathcal{T}$. We introduce the *Ringel dual* of bi-admissible echelon $\mathsf{R}\mathcal{E}$. It is a bi-admissible echelon of the same length n and the same top $\mathcal{T}$ defined in the following way. At (a,b)-th vertex, we put $\mathcal{E}_{n+1-b,n+1-a}$. The functors are defined to be $I^{\mathsf{R}\mathcal{E}}_{ab} := (J^{\mathcal{E}}_{n+1-b,n-a})^{\mathsf{R}}$ and $J^{\mathsf{R}\mathcal{R}}_{ab} := (I^{\mathcal{E}}_{n-b,n+1-a})^{\mathsf{R}}$. This diagram become a bi-admissible echelon.

Below we give a picture of the Ringel dual $\mathsf{R}\mathcal{E}$ and the double Ringel dual $\mathsf{R}^2\mathcal{E}$ of a bi-admissible echelon $\mathcal{E}$ of length 3.

$$\mathsf{R}\mathcal{E}: \begin{tikzcd} \mathcal{E}_{33} \arrow[r, "J^{\mathsf{R}}_{23}"] & \mathcal{E}_{23} \arrow[r, "J^{\mathsf{R}}_{13}"] \arrow[d, "I^{\mathsf{R}}_{22}"'] & \mathcal{E}_{13} = \mathcal{T} \arrow[d, "I^{\mathsf{R}}_{12}"] \\ & \mathcal{E}_{22} \arrow[r, "J^{\mathsf{R}}_{12}"'] & \mathcal{E}_{12} \arrow[d, "I^{\mathsf{R}}_{11}"] \\ & & \mathcal{E}_{11} \end{tikzcd}, \quad \mathsf{R}^2\mathcal{E}: \begin{tikzcd} \mathcal{E}_{11} \arrow[r, "I^{\mathsf{R}^2}_{11}"] & \mathcal{E}_{12} \arrow[r, "I^{\mathsf{R}^2}_{12}"] \arrow[d, "J^{\mathsf{R}^2}_{12}"'] & \mathcal{E}_{13} = \mathcal{T} \arrow[d, "J^{\mathsf{R}^2}_{13}"] \\ & \mathcal{E}_{22} \arrow[r, "I^{\mathsf{R}^2}_{22}"'] & \mathcal{E}_{23} \arrow[d, "J^{\mathsf{R}^2}_{23}"] \\ & & \mathcal{E}_{33} \end{tikzcd}$$

Definition 4. Let $\mathcal{F}$ be a bi-admissible filtration of $\mathcal{T}$. The *Ringel dual* $\mathsf{R}\mathcal{F}$ is defined to be

$$\mathsf{R}\mathcal{F} := \mathsf{FRE}(\mathcal{F}).$$

Remark 5. Despite of the name, in general, the double Ringel dual $\mathsf{R}^2\mathcal{F}$ has no relationship with the original $\mathcal{F}$.

2.1. **A bi-admissible filtration and associated recollements.** A recollement of the following form will be denoted as $\mathcal{R} : \mathcal{T}' \overset{i}{\equiv} \mathcal{T} \overset{j}{\equiv} \mathcal{T}''$.

$$\begin{tikzcd} \mathcal{T}' \arrow[d, "i_*"] \\ \mathcal{T} \arrow[u, bend left=60, "i^*"] \arrow[u, bend right=60, "i^!"'] \arrow[d, "j^*"] \\ \mathcal{T}'' \arrow[u, bend left=60, "j_!"] \arrow[u, bend right=60, "j_*"'] \end{tikzcd}$$

Lemma 6. *A bi-admissible embedding $\mathcal{T}' \overset{I}{\hookrightarrow} \mathcal{T}$ and a recollement $\mathcal{R} : \mathcal{T}' \overset{i}{\equiv} \mathcal{T} \overset{j}{\equiv} \mathcal{T}''$ such that $i_* = I$ are the same thing up to equivalences.*

By Lemma 6, a bi-admissible filtration $\mathcal{F}$ of length n induces $n-1$ recollements $\mathcal{F}_a$

$$\mathcal{F}_a : \mathcal{F}_a \overset{i^{\mathcal{F}}_a}{\equiv} \mathcal{F}_{a+1} \overset{j^{\mathcal{F}}_{a+1}}{\equiv} \mathcal{G}_{a+1}$$

with $i^{\mathcal{F}}_{a*} := I_a$. By convention, we set $\mathcal{G}_1 := \mathcal{F}_1$. We call the recollement $\mathcal{F}_a$ the a-th filter of $\mathcal{F}$. We set $\tilde{i}_{a*} := i_{n*}i_{n-1*}\cdots i_{a*} : \mathcal{F}_a \to \mathcal{F}_n$ and via this functor we consider $\mathcal{F}_a$ as a subcategory of $\mathcal{F}_n$. We also introduce notations for functors $\mathcal{G}_a \to \mathcal{F}_n$ as follows: $\tilde{j}_{a!} := \tilde{i}_{a*}j_{a!}, \tilde{j}_{a*} := \tilde{i}_{a*}j_{a*}$ for $a = 2,\dots,n$ and $\tilde{j}_{1!} := \tilde{i}_{1*}, \tilde{j}_{1*} := \tilde{i}_{1*}$.

Note that each filter $(\mathsf{R}\mathcal{F})_a$ is of the form

$$\mathsf{R}\mathcal{F}_a : \mathcal{F}_n/\mathcal{F}_{n-a} \overset{i_a^{\mathsf{R}\mathcal{F}}}{\equiv} \mathcal{F}_n/\mathcal{F}_{n-a-1} \overset{j_{a+1}^{\mathsf{R}\mathcal{F}}}{\equiv} \mathcal{G}_{n-a}$$

In particular, the graded quotients of $\mathsf{R}\mathcal{F}$ are $\{\mathcal{G}_a \mid a = 1, \dots, n\}$ as sets, but ordered in the reverse order.

2.2. **Recollement filtration and t-structures.** Let $\mathcal{R} : \mathcal{T}' \overset{i}{\equiv} \mathcal{T} \overset{j}{\equiv} \mathcal{T}''$ be a recollement. Assume that t -structures T' and T'' in $\mathcal{T}'$ and $\mathcal{T}''$ respectively are given. Recall that by Beilinson-Bernstein-Deligne ([1]), the pair $\mathsf{T} = (\mathsf{T}^{\leq 0}, \mathsf{T}^{\geq 0})$ of full subcategories defined in the followings way is a t-structure in $\mathcal{T}$.

$$\mathsf{T}^{\leq 0} = \{X \in \mathcal{T} \mid j^*X \in \mathsf{T}''^{\leq 0}, i^*X \in \mathsf{T}'^{\leq 0}\},$$
$$\mathsf{T}^{\geq 0} = \{X \in \mathcal{T} \mid j^*X \in \mathsf{T}''^{\geq 0}, i^!X \in \mathsf{T}'^{\geq 0}\}.$$

We set $\mathrm{Ind}_{\mathcal{R}}(\mathsf{T}', \mathsf{T}'') := \mathsf{T}$ (the Beilinson-Bernstein-Deligne induction).

Let $\mathcal{F}$ be a bi-admissible filtration of $\mathcal{T}$. Assume that t-structures T_a in the graded quotients $\mathcal{G}_a$ of $\mathcal{F}$ are given. Then we set

$$\mathrm{Ind}_{\mathcal{F}}(\mathsf{T}_a)_{a=1}^n := \mathrm{Ind}_{\mathcal{F}_n}(\cdots \mathrm{Ind}_{\mathcal{F}_2}(\mathrm{Ind}_{\mathcal{F}_1}(\mathsf{T}_0, \mathsf{T}_1), \mathsf{T}_2) \dots, \mathsf{T}_n).$$

Lemma 7. *We denote by S, S_a the Serre functors of $\mathcal{T}$ and $\mathcal{G}_a$.*

$$\mathrm{Ind}_{\mathsf{R}^2(\mathcal{F})}(\mathsf{T}_a)_{a=1}^n = \mathsf{S}(\mathrm{Ind}_{\mathcal{F}}(\mathsf{S}_a^{-1}\mathsf{T}_a)_{a=1}^n).$$

Let $p, q \in \mathbb{Z}$ with $q \neq 0$. Recall that a triangulated category $\mathcal{T}$ with a Serre functor S is called p/q-Calabi-Yau if $\mathsf{S}^q = [p]$.

Corollary 8. *Assume that every graded quotient $\mathcal{G}_a$ is p/q-Calabi-Yau. Then we have*

$$\mathrm{Ind}_{\mathsf{R}^{2q}(\mathcal{F})}(\mathsf{T}_a)_{a=1}^n = \mathsf{S}^q[-p](\mathrm{Ind}_{\mathcal{F}}(\mathsf{T}_a)_{a=1}^n)$$

The following is the $0/1$-Calabi-Yau case.

Corollary 9. *Assume that the identity functor of graded quotient $\mathcal{G}_a$ is a Serre functor. Then we have*

$$\mathrm{Ind}_{\mathsf{R}^2(\mathcal{F})}(\mathsf{T}_a)_{a=1}^n = \mathsf{S}(\mathrm{Ind}_{\mathcal{F}}(\mathsf{T}_a)_{a=1}^n)$$

3. Relationship to usual Ringel duality for quasi-hereditary algebras

Let $\Lambda = (\Lambda, <)$ be a quasi-hereditary algebra. We denote by $\{e_a\}_{a=1}^n$ the idempotent elements. It has been known by Cline-Parshall-Scott [3] that the bounded derived category $\mathcal{T} = \mathcal{D}^{\mathrm{b}}(\Lambda)$ of a quasi-hereditary algebra Λ has a bi-admissible filtration $\mathcal{F}$ of length n whose graded quotients $\mathcal{G}_a$ is the bounded derived category $\mathcal{D}^{\mathrm{b}}(\Gamma_a)$ of the division algebra Γ_a corresponding to the idempotent e_a.

The following theorem justify the name Ringel duality for bi-admissible filtrations.

Theorem 10. *The Ringel dual $\mathsf{R}(\mathcal{F})$ is the bi-admissible filtration of $\mathcal{T}$ corresponding to the Ringel dual $\mathsf{R}(\Lambda)$.*

4. Generalizing Ringel duality for (DG-)algebras

4.1. **Generalizing Ringel duality for (DG-)algebras.** Using the result by Koenig-Yang, we can define a Ringel dual for algebras Λ with appropriate idempotents or appropriate bi-admissible filtration $\mathcal{F}$. The Ringel dual $\mathsf{R}_{\mathcal{F}}(\Lambda)$ possibly become a dg-algebra.

Let Λ be a finite dimensional algebra and $\mathcal{F}$ a bi-admissible filtration of length n of the derived category $\mathcal{T} := \mathcal{D}^{\mathrm{b}}(\Lambda)$ of Λ. We assume that $\mathcal{F}$ satisfies the following conditions:

(1) Assume that each graded quotient $\mathcal{G}_a$, $(a = 1, \dots, n)$ is the derived category $\mathcal{D}^{\mathrm{b}}(\Gamma_a)$ of some finite dimensional algebra Γ_a.
    We denote the standard t-structure of $\mathcal{G}_a = \mathcal{D}^{\mathrm{b}}(\Gamma_a)$ by St_a.
(2) We assume that the induced t-structure $\mathrm{Ind}_{\mathcal{F}}(\mathsf{St}_a)_{a=1}^n$ is the standard t-structure St_Λ of $\mathcal{T} = \mathcal{D}^{\mathrm{b}}(\Lambda)$.

The induced t-structure $\mathrm{Ind}_{\mathsf{R}(\mathcal{F})}(\mathsf{St}_{n+1-a})_{a=1}^n$ is a bounded t-structure whose heart is a length abelian category [8]. Therefore by [5] , there is a silting object $S \in \mathcal{T}$ corresponds to the induced t-structure.

Definition 11. We define the *Ringel dual* $\mathsf{R}_{\mathcal{F}}(\Lambda)$ of Λ with respect to a bi-admissible filtration $\mathcal{F}$ satisfying above assumption to be the endomorphism dg-algebra of the silting object S:

$$\mathsf{R}_{\mathcal{F}}(\Lambda) := \mathbb{R}\mathrm{End}_\Lambda(S).$$

Note that the induced t-structure $\mathrm{Ind}_{\mathsf{R}(\mathcal{F})}(\mathsf{St}_{n+1-a})_{a=1}^n$ corresponds to the standard t-structure $\mathsf{St}_{\mathsf{R}_{\mathcal{F}}(\Lambda)}$ of the derived category $\mathcal{D}^{\mathrm{b}}(\mathsf{R}_{\mathcal{F}}(\Lambda))$ under the equivalence $\mathbb{R}\mathrm{Hom}_\Lambda(S, -) : \mathcal{T} \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\mathsf{R}_{\mathcal{F}}(\Lambda))$. We denote the induced functor by $\mathsf{RD}_{\mathcal{F}} : \mathcal{D}^{\mathrm{b}}(\Lambda) \to \mathcal{D}^{\mathrm{b}}(\mathsf{R}_{\mathcal{F}}\Lambda)$.

Theorem 12. *Let p, q be integers such that $q \neq 0$. Assume that every Γ_a is p/q-Calabi-Yau. Then we have $\mathsf{R}^{2q}(\Lambda) \cong \Lambda$. Moreover if we identify these two algebras, then we have $\mathsf{RD}^{2q} = \mathsf{S}^{-q}[p]$. More precisely,*

$$\mathsf{S}^{-q}[p] : \mathcal{D}^{\mathrm{b}}(\Lambda) \xrightarrow{\mathsf{RD}_{\mathcal{F}}} \mathcal{D}^{\mathrm{b}}(\mathsf{R}_{\mathcal{F}}\Lambda) \xrightarrow{\mathsf{RD}_{\mathcal{F}}} \cdots \to \mathcal{D}^{\mathrm{b}}(\mathsf{R}_{\mathcal{F}}^{2q-1}\Lambda) \xrightarrow{\mathsf{RD}_{\mathcal{F}}} \mathcal{D}^{\mathrm{b}}(\mathsf{R}_{\mathcal{F}}^{2q}\Lambda) \cong \mathcal{D}^{\mathrm{b}}(\Lambda)$$

A. Chan suggested to me that in the situation of above theorem it might be better call $\mathsf{R}_{\mathcal{F}}^q\Lambda$ the Ringel dual of Λ, rather than $\mathsf{R}_{\mathcal{F}}\Lambda$.

4.2. **Generalized Ringel dual of a path algebra of $\mathbf{A}_N$ quiver.** We compute the Ringel dual of a path algebra of A_N quiver.

Let $a_1, a_2, \dots, a_n$ be natural numbers. Set $A_0 = 1$ and $A_i := a_1 + a_2 + \cdots + a_i$ for $i = 1, 2, \dots, n$. Let $\Lambda := \mathbf{k}[1 \to 2 \to \cdots \to (A_n - 1) \to A_n]$ be a path algebra of A_{A_n}-quiver and e'_j the idempotent element corresponds to the vertex $1 \leq j \leq A_n$. Set $e_i := e'_{A_{i-1}+1} + e'_{A_{i-1}+2} + \cdots + e'_{A_i}$ and $\Gamma_i := \mathbf{k}[(A_{i-1} + 1) \to (A_{i-1} + 2) \to \cdots \to A_i]$ for $i = 1, \dots, n$. Then the sequence $\{e_i\}_{i=1}^n$ induces the filtration $\mathcal{F}$ of recollement of the derived category $\mathcal{D}^b(\mathrm{mod}\,\Lambda)$ whose graded quotients are naturally isomorphic to the derived category $\mathcal{D}^b(\mathrm{mod}\,\Gamma_i)$. Moreover the filtration $\mathcal{F}$ satisfies the assumptions (1) and (2) above. We compute the Ringel dual $\mathsf{R}_{\mathcal{F}}(\Lambda)$ of Λ with respect to the filtration $\mathcal{F}$.

Proposition 13. *The Ringel dual* $\mathsf{R}_{\mathcal{F}}(\Lambda)$ *is given by the following quiver with relations.*

$$
\begin{tikzcd}
1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & a_n - 1 \arrow[r] \arrow[dr, dashed, no head] & a_n \arrow[d] \\
1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & a_{n-1} - 1 \arrow[r] \arrow[dr, dashed, no head] & a_{n-1} \arrow[d] \\
1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & a_{n-2} - 1 \arrow[r] \arrow[dr, dashed, no head] & a_{n-2} \arrow[d] \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & a_1 - 1 \arrow[r] & a_1
\end{tikzcd}
$$

Example 14. If $a_1 = 3, a_2 = 2, a_3 = 4$, then the quiver with relations is

$$
\begin{tikzcd}
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[dr, dashed, no head] & \bullet \arrow[d] \\
 & & \bullet \arrow[r] \arrow[dr, dashed, no head] & \bullet \arrow[d] \\
 & \bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}
$$

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Hiroyuki Minamoto
Department of Mathematics and Information Sciences
Graduate School of Science
Osaka Prefecture University
Sakai Nakamozu, Osaka 599-8531 JAPAN
Email: minamoto@mi.s.osakafu-u.ac.jp