m-KOSZUL AS-REGULAR ALGEBRAS AND TWISTED SUPERPOTENTIALS

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Abstract. In noncommutative algebraic geometry, AS-regular algebras are the most important class of algebras to study. If S is an m-Koszul AS-regular algebra, then it was observed by Dubois-Violette that S is determined by a twisted superpotential. In this article, we will see that such a twisted superpotential is uniquely determined by S up to non-zero scalar multiples and plays a crucial role in studying S. In particular, using the twisted superpotential (uniquely) determined by S, we will see that we can compute (1) the Nakayama automorphism of S, (2) the group of graded algebra automorphisms of S, and (3) the homological determinant of a graded algebra automorphism of S.

1. Motivation

Throughout this article, we fix an algebraically closed field k of characteristic 0, and let $S = k\langle x_1, \dots, x_n\rangle/(f_1, \dots, f_r)$ be a finitely presented connected graded algebra (that is, $\deg x_i \geq 1$ for all i and $f_j \in k\langle x_1, \dots, x_n\rangle$ are homogeneous for all j). In this case, $\mathfrak{m} := (x_1, \dots, x_n)$ is the unique homogeneous maximal ideal of S and we view $k = S/\mathfrak{m}$ as a graded right S-module. An AS-regular algebra defined below is the most important algebra to study in noncommutative algebraic geometry.

Definition 1. [1] We say that S is a d-dimensional AS-regular (AS-Gorenstein, resp.) algebra if

(1) $\operatorname{gldim} S = d < \infty$ ($\operatorname{id} S = d < \infty$, resp.), and
(2) $\operatorname{Ext}^i_S(k, S) \cong \begin{cases} k & \text{if } i = d, \\ 0 & \text{if } i \neq d. \end{cases}$

If S is an AS-regular algebra, then there exists a unique graded algebra automorphism $\nu_S \in \operatorname{Aut} S$ such that $D\operatorname{H}^d_{\mathfrak{m}}(S) \cong S_{\nu_S}$ as S-S bimodules where $D(-) := \operatorname{Hom}_k(-, k)$, $\operatorname{H}^d_{\mathfrak{m}}(S) := \lim_{i\to\infty} \operatorname{Ext}^d_S(S/\mathfrak{m}^i, S)$ is the d-th local cohomology of S with respect to $\mathfrak{m}$, and $S_{\nu_S} = S$ as graded vector spaces with the new bimodule structure $a * x * b := ax\nu_S(b)$. We call ν_S the Nakayama automorphism of S.

Definition 2. We say that S is Calabi-Yau if

(1) S has a resolution of length d by finitely generated projective right S^{e}-modules, and
(2) $\operatorname{Ext}^i_{S^{\mathrm{e}}}(S, S^{\mathrm{e}}) \cong \begin{cases} S & \text{if } i = d, \\ 0 & \text{if } i \neq d \end{cases}$ as S-S bimodules,

where $S^{\mathrm{e}} := S^{\mathrm{op}} \otimes_k S$ is the enveloping algebra of S.

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The Nakayama automorphism is important by the following theorem.

Theorem 3. [7] *Let S be a noetherian AS-regular algebra. Then S is Calabi-Yau if and only if ν_S is the identity.*

If S is an AS-regular algebra, then, for a graded algebra automorphism $\sigma \in \operatorname{Aut} S$, there exists $\lambda \in k \setminus \{0\}$ such that the diagram

$$\begin{array}{ccc} \mathrm{H}^d_{\mathfrak{m}}(S) & \xrightarrow{\mathrm{H}^d_{\mathfrak{m}}(\sigma)} & \mathrm{H}^d_{\mathfrak{m}}(S) \\ \cong\downarrow & & \downarrow\cong \\ D(S) & \xrightarrow{\lambda D(\sigma^{-1})} & D(S) \end{array}$$

commutes. We call $\operatorname{hdet}\sigma := \lambda$ the homological determinant of σ. The homological determinant plays an essential role in the invariant theory for AS-regular algebras by the following theorem.

Theorem 4. [5] *Let S be a noetherian AS-regular algebra and $G \leq \operatorname{Aut} S$ a finite subgroup. If $\operatorname{hdet}\sigma = 1$ for every $\sigma \in G$, then the invariant subalgebra S^G of S by G is a noetherian AS-Gorenstein algebra.*

Although the Nakayama automorphism and the homological determinant are essential ingredients in noncommutative algebraic geometry, they are rather mysterious and certainly difficult to compute from their definitions. The purpose of this article is to give easy ways to compute them using twisted superpotentials.

2. Twisted Superpotentials

From now on, we tacitly assume that $\deg x_i = 1$ for all i so that $V := kx_1 + \cdots + kx_n = S_1$ and $S = T(V)/I$ where $T(V)$ is the tensor algebra of V over k and I is a homogeneous ideal of $T(V)$. We define a linear map $\phi : V^{\otimes \ell} \to V^{\otimes \ell}$ by

$$\phi(v_1 \otimes v_2 \otimes \cdots \otimes v_\ell) = v_\ell \otimes v_1 \otimes \cdots \otimes v_{\ell-1}.$$

Definition 5. Let $\mathsf{w} \in V^{\otimes \ell}$.

(1) We call w a twisted superpotential if there exists $\sigma \in \mathrm{GL}_n(k)$ such that

$$(\sigma \otimes \mathrm{id}^{\otimes \ell-1})\phi(\mathsf{w}) = \mathsf{w}.$$

(2) We call w a superpotential if $\phi(\mathsf{w}) = \mathsf{w}$, that is, if w is a twisted superpotential with $\sigma = \mathrm{id}$.

For every $\mathsf{w} \in V^{\otimes \ell}$, there exist unique $\mathsf{w}_i \in V^{\otimes \ell - 1}$ such that $\mathsf{w} = \sum_{i=1}^n x_i \otimes \mathsf{w}_i$. We define the left partial derivative of w with respect to x_i by $\partial_{x_i}\mathsf{w} := \mathsf{w}_i$. The right partial derivatives $\mathsf{w}\partial_{x_i}$ are defined in an obvious way. We can iterate this operation to define higher order partial derivatives such as $\partial_{x_i x_j}\mathsf{w} := \partial_{x_i}(\partial_{x_j}\mathsf{w})$. The (higher order) left derivation quotient algebras of w are defined as follows:

$$\begin{aligned}\mathcal{D}(\mathsf{w},0)&=k\langle x_1,\dots,x_n\rangle/(\mathsf{w})\\ \mathcal{D}(\mathsf{w},1)&=k\langle x_1,\dots,x_n\rangle/(\partial_{x_i}\mathsf{w})_{1\le i\le n}\\ \mathcal{D}(\mathsf{w},2)&=k\langle x_1,\dots,x_n\rangle/(\partial_{x_ix_j}\mathsf{w})_{1\le i,j\le n}\\ &\vdots\end{aligned}$$

Example 6. If $\mathsf{w}=\alpha x^2+\beta xy+\gamma yx+\delta y^2$, then $\phi(\mathsf{w})=\alpha x^2+\beta yx+\gamma xy+\delta y^2$, so w is a superpotential if and only if $\beta=\gamma$. Since $\mathsf{w}=x(\alpha x+\beta y)+y(\gamma x+\delta y)$, $\partial_x\mathsf{w}=\alpha x+\beta y, \partial_y\mathsf{w}=\gamma x+\delta y$. On the other hand, since $\mathsf{w}=(\alpha x+\gamma y)x+(\beta x+\delta y)y$, $\mathsf{w}\partial_x=\alpha x+\gamma y, \mathsf{w}\partial_y=\beta x+\delta y$.

Definition 7. [2] We say that $S=k\langle x_1.\dots,x_n\rangle/(f_1,\dots,f_r)$ is m-Koszul if
1. $\deg f_j=m$ for all j, and
2. $\mathrm{Ext}^i_S(k,k)$ are concentrated in one degree for all i.

Every noetherian AS-regular algebra (generated in degree 1) up to dimension 3 is known to be m-Koszul for $m=2,3$. In the theorem below, the existence of a superpotential was proved in [4] and the uniqueness of a superpotential was proved in [6].

Theorem 8. [4], [6] *For every m-Koszul AS-regular algebra, there exists a unique twisted superpotential $\mathsf{w}_S\in V^{\otimes\ell}$ up to non-zero scalar multiples such that $S=\mathcal{D}(\mathsf{w}_S,\ell-m)$.*

By the uniqueness of the superpotential w_S, we expect to derive various properties of S from w_S, as in the subsequent sections.

3. Nakayama Automorphisms

For $\mathsf{w}\in V^{\otimes\ell}$, we define the $n\times n$ matrix $M(\mathsf{w}):=(\partial_{x_i}\mathsf{w}\partial_{x_j})$ with entries in $V^{\otimes\ell-2}$.

Theorem 9. [6] *Let $\mathsf{w}\in V^{\otimes\ell}$ and $\mathbf{x}=(x_1,\dots,x_n)^t$ a column vector with entries in V.*
1. *w is a twisted superpotential if and only if there exists $Q(\mathsf{w})\in\mathrm{GL}_n(k)$ such that $(\mathbf{x}^tM(\mathsf{w}))^t=Q(\mathsf{w})M(\mathsf{w})\mathbf{x}$. In this case, w is a superpotential if and only if $Q(\mathsf{w})$ is the identity matrix.*
2. *If S is an m-Koszul d-dimensional AS-regular algebra, then $\nu_S=(-1)^{d+1}(Q(\mathsf{w}_S)^{-1})^t$.*

Remark 10. Let S be a noetherian m-Koszul d-dimensional AS-regular algebra. By Theorem 9, if d is odd, then S is Calabi-Yau if and only if w_S is a superpotential, but if d is even, then this is false. (This explains a mystery in [3].)

Example 11. It is known that S is a noetherian 2-dimensional AS-regular algebra if and only if $\mathsf{w}_S=\alpha x^2+\beta xy+\gamma yx+\delta y^2$ such that $\alpha\delta\neq\beta\gamma$. In this case, it is easy to compute $M(\mathsf{w}_S)=\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}$, so

$$\begin{aligned}\nu_S&=-(Q(\mathsf{w}_S)^{-1})^t=-((M(\mathsf{w}_S)^tM(\mathsf{w}_S)^{-1})^{-1})^t=-M(\mathsf{w}_S)^{-1}M(\mathsf{w}_S)^t\\ &=-\frac{1}{\alpha\delta-\beta\gamma}\begin{pmatrix}\alpha\delta-\gamma^2&-\alpha(\beta-\gamma)\\ (\beta-\gamma)\delta&\alpha\delta-\beta^2\end{pmatrix}.\end{aligned}$$

It follows that S is Calabi-Yau if and only if $\alpha=\delta=\beta+\gamma=0$.

4. Homological Determinants

Since $S = T(V)/I$, every graded algebra automorphism $\sigma \in \operatorname{Aut} S$ restricts to $\sigma|_V \in \mathrm{GL}(V)$, but not conversely in general. In fact, it is not easy to compute $\operatorname{Aut} S$. However, if S is an m-Koszul AS-regular algebra, then we can compute $\operatorname{Aut} S$ and the homological determinant of $\sigma \in \operatorname{Aut} S$ using w_S.

Theorem 12. [6] *Let S be an m-Koszul AS-regular algebra.*

(1) *$\sigma \in \mathrm{GL}(V)$ extends to a graded algebra automorphism $\overline{\sigma} \in \operatorname{Aut} S$ if and only if $\sigma^{\otimes \ell}(\mathsf{w}_S) = \lambda \mathsf{w}_S$ for some $\lambda \in k \setminus \{0\}$.*

(2) *If $\sigma \in \operatorname{Aut} S$, then $\sigma|_V^{\otimes \ell}(\mathsf{w}_S) = (\operatorname{hdet} \sigma)\mathsf{w}_S$.*

If $S = k[x_1, \dots, x_n]$ is a commutative polynomial algebra (which is a typical example of a noetherian AS-regular Calabi-Yau algebra) and $\sigma \in \operatorname{Aut} S$, then it is known that $\operatorname{hdet} \sigma = \det \sigma|_V$. This is often the case but not always. For example, if S is a noetherian 2-dimensional AS-regular algebra, then it is known that there exists $\sigma \in \operatorname{Aut} S$ such that $\operatorname{hdet} \sigma \neq \det \sigma|_V$ if and only if $\mathsf{w}_S \in \operatorname{Sym}^2 V$. We have a similar result in dimension 3.

Theorem 13. [6] *Let S be a noetherian 3-dimensional quadratic AS-regular algebra. Then there exists $\sigma \in \operatorname{Aut} S$ such that $\operatorname{hdet} \sigma \neq \det \sigma|_V$ if and only if $\mathsf{w}_S \in \operatorname{Sym}^3 V$.*

One direction of Theorem 13 can be proved as follows. In the above setting, if $\mathsf{w}_S \in \operatorname{Sym}^3 V$, then

$$\mathsf{w}_S = xyz + yzx + zxy + xzy + yxz + zyx + \alpha x^3 + \beta y^3 + \gamma z^3.$$

By change of the basis, we may assume that $\alpha = \beta$. If $\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_3(k)$, then $\sigma^{\otimes 3}(\mathsf{w}_S) = \mathsf{w}_S$. It follows that σ extends to a graded algebra automorphism $\overline{\sigma} \in \operatorname{Aut} S$ and $\operatorname{hdet} \overline{\sigma} = 1 \neq -1 = \det \sigma$.

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