

# INTRODUCTION TO QUIVER VARIETIES — FOR RING AND REPRESENTATION THEORISTS

HIRAKU NAKAJIMA

ABSTRACT. We review the definition of quiver varieties and their relation to representation theory of Kac-Moody Lie algebras. Target readers are ring and representation theorists. We emphasize important roles of first extension groups of the preprojective algebra associated with a quiver.

## 1. INTRODUCTION

This is a review on quiver varieties written for the proceeding of 49th Symposium on Ring Theory and Representation Theory at Osaka Prefecture University, 2016 Summer, based on my two lectures. Quiver varieties are spaces parametrising representations of preprojective algebras associated with a quiver, hence they are closely related to Ring Theory and Representation Theory. This is the reason why I was invited to give lectures, even though my main research interest is geometric representation theory.

The purpose of this review is to explain the definition of quiver varieties and the main result in [Na94, Na98], which is 20 years old. Why do I write a review of such an old result ? There exist several reviews of quiver varieties already. An earlier review with the same target readers is [Na96]. There are other reviews [Sc08, Gi12], and also a book [Ki16].

Besides shortest among existing reviews, this one has a special feature: I put emphasis on the complex (2.5), which has been used at various places in the theory of quiver varieties. It is a familiar complex in representation theory, as it computes homomorphisms and  $\text{Ext}^1$  between (framed) representations of the preprojective algebra.

Importance of  $\text{Ext}^1$  is clear to ring and representation theorists. A purpose of this review is to explain its importance for quiver varieties. Quiver varieties themselves could be loosely viewed as *nonlinear* analog of self- $\text{Ext}^1$  as their tangent spaces are nothing but  $\text{Ext}^1$  of modules with themselves. (See at the end of §2.1.) A particularly nice feature of (2.5) for the case of tangent spaces is the vanishing of the first and third cohomology groups. This follows from the stability condition used in the definition of quiver varieties. It implies the smoothness of quiver varieties.

The complex (2.5) where one of representation corresponds to a simple representation  $S_i$  for a vertex  $i$  is also important. See §4.5. The stability condition implies that the first cohomology group vanishes, hence the difference of dimensions of the second and third cohomology groups is the Euler characteristic of the complex. This simple observation plays an important role. The complex also appears in a definition of Kashiwara crystal

---

The paper is in a final form and no version of it will be submitted for publication elsewhere.

structure on the set of irreducible components of lagrangian subvarieties in quiver varieties. See §4.6.

I will not list earlier references which I studied before writing [Na94, Na98]. So readers who pay attention on history should read original papers. But two papers [Ri90, Lu90a] were so fundamental, let me recall how I encountered them. In 1989 Summer I introduced moduli spaces  $\mathfrak{M}(V, W)$  of (framed) representations of the preprojective algebra of an affine  $ADE$  quiver  $Q = (Q_0, Q_1)$  with dimension vector  $V$  and framing vector  $W$  with Kronheimer [KN90]. This construction had an origin in the gauge theory. Hence I thought that they are important spaces, and I was interested in symplectic geometry, topology, etc, of  $\mathfrak{M}(V, W)$  as a geometer.

In 1990 Summer I heard Lusztig's plenary talk at ICM Kyoto, explaining his construction [Lu90a] of the canonical base of the upper triangular subalgebra  $\mathbf{U}^-$  of the quantized enveloping algebra  $\mathbf{U} = \mathbf{U}_q(\mathfrak{g})$ , built on an earlier result by Ringel [Ri90].

Since Ringel and Lusztig's constructions were based on fields which were not familiar to me at that time, it took several years until I realized that the direct sum of homology groups of  $\mathfrak{M}(V, W)$  is a representation of the Kac-Moody Lie algebra  $\mathfrak{g}$  associated with the quiver  $Q$ , as variants of their construction [Na94, Na98]. This makes sense for any quiver, hence I named  $\mathfrak{M}(V, W)$  *quiver varieties*, and started to study further structures of  $\mathfrak{M}(V, W)$ . Rather unexpectedly, quiver varieties have lots of structures, and they are still actively studied by various people even now.

**Acknowledgment.** I thank the organizers of the symposium for invitation.

## 2. NOTATION AND BASIC DEFINITIONS

**2.1. Preprojective algebras and extension groups.** Let  $Q = (Q_0, Q_1)$  be a quiver, where  $Q_0$  is the set of vertices, and  $Q_1$  is the set of oriented edges. We always assume  $Q$  is finite. Let  $o(h)$ ,  $i(h)$  denote the outgoing and incoming vertices of an edge  $h$ . For  $h \in Q_1$ , we consider an edge with opposite orientation and denote it by  $\bar{h}$ , hence  $o(\bar{h}) = i(h)$ ,  $i(\bar{h}) = o(h)$ . We add  $\bar{Q}_1 = \{\bar{h} \mid h \in Q_1\}$  to  $Q_1$ , and consider the *doubled quiver*  $Q^{\text{dbl}} = (Q_0, Q_1 \sqcup \bar{Q}_1)$ . We denote  $Q_1 \sqcup \bar{Q}_1$  by  $Q_1^{\text{dbl}}$ . We extend  $\bar{\cdot} : Q_1 \rightarrow \bar{Q}_1$  to  $Q_1^{\text{dbl}}$  so that  $\bar{\bar{h}} = h$ .

Let  $V = \bigoplus_{i \in Q_0} V_i$  be a finite dimensional complex  $Q_0$ -graded vector space. Its dimension vector  $(\dim V_i)_{i \in Q_0} \in \mathbb{Z}_{\geq 0}^{Q_0}$  is denoted simply by  $\dim V$ . We introduce a vector space

$$\mathbf{N}(V) = \bigoplus_{h \in Q_1} \text{Hom}(V_{o(h)}, V_{i(h)}), \quad G_V = \prod_{i \in Q_0} \text{GL}(V_i).$$

An element in  $\mathbf{N}(V)$  is denoted by  $B = \bigoplus_{h \in Q_1} B_h$  or  $(B_h)_{h \in Q_1}$ , where  $B_h$  is the component in  $\text{Hom}(V_{o(h)}, V_{i(h)})$ . Similarly an element in  $G_V$  is denoted by  $g = \prod_{i \in Q_0} g_i = (g_i)_{i \in Q_0}$ .

We have a set-theoretical bijection

$$\left\{ \begin{array}{l} \text{isomorphism classes of representations of } Q \text{ whose dimension} \\ \text{vector is } \dim V \end{array} \right\} \longleftrightarrow \mathbf{N}(V)/G_V.$$

Here  $G_V$  acts on  $\mathbf{N}(V)$  by conjugation. By abuse of terminology a point  $B \in \mathbf{N}(V)$  is often called a representation of  $Q$ .

We consider the cotangent space to  $\mathbf{N}(V)$ :

$$\mathbf{M}(V) = \mathbf{N}(V) \oplus \mathbf{N}(V)^* = \bigoplus_{h \in Q_1^{\text{dbl}}} \text{Hom}(V_{o(h)}, V_{i(h)}).$$

The action of  $G_V$  on  $\mathbf{M}(V)$  is defined also by conjugation. It preserves the symplectic form on  $\mathbf{M}(V)$  given by the natural pairing. We consider the associated *moment map*

$$\mu: \mathbf{M}(V) \rightarrow \bigoplus_{i \in Q_0} \text{End}(V_i); \quad \bigoplus_{h \in Q_1^{\text{dbl}}} B_h \mapsto \bigoplus_{i \in Q_0} \sum_{\substack{h \in Q_1^{\text{dbl}} \\ i(h)=i}} \varepsilon(h) B_h B_{\bar{h}},$$

where  $\varepsilon(h) = 1$  if  $h \in Q_1$ ,  $-1$  if  $h \in \bar{Q}_1$ .

Note that  $\bigoplus_i \text{End}(V_i)$  is the Lie algebra of the group  $G_V$ .

The moment map has its origin in symplectic geometry: the quotient space  $\mu^{-1}(0)/G_V$  is the cotangent bundle  $T^*(\mathbf{N}(V)/G_V)$  of  $\mathbf{N}(V)/G_V$ :  $(B_h)_{h \in \bar{Q}_1}$  is a cotangent vector, and the equation  $\mu = 0$  means it vanishes to the tangent direction to  $G_V$ -orbits:

$$\begin{aligned} (B_{\bar{h}})_{\bar{h} \in \bar{Q}_1} \perp T(G_V \cdot (B_h)_{h \in Q_1}) \\ \iff \sum_{h \in Q_1} \text{tr}(B_{\bar{h}}(\xi_{i(h)} B_h - B_h \xi_{o(h)})) = 0 \quad \forall (\xi_i)_i \in \bigoplus_{i \in Q_0} \text{End}(V_i) \\ \iff \sum_{h \in Q_1, i(h)=i} B_h B_{\bar{h}} - \sum_{h \in Q_1, o(h)=i} B_{\bar{h}} B_h = 0. \end{aligned}$$

But this must be understood with care, as  $\mathbf{N}(V)/G_V$  is not a manifold, nor even a Hausdorff space, in general. In the next section we introduce a modification of the quotient  $\mu^{-1}(0)/G_V$  (we also add framing), which is a smooth algebraic variety. But it is *not* a cotangent bundle, nor a vector bundle over another manifold. It is because a similar modification of  $\mathbf{N}(V)/G_V$  is usually smaller or quite often  $\emptyset$ , and its cotangent bundle is an open subset of  $\mu^{-1}(0)/G_V$ . Here the cotangent bundle of an empty set is understood as an empty set.

Let us also note that  $\mu = 0$  is the defining relation of the *preprojective algebra*  $\Pi(Q)$ , introduced by Gelfand-Ponomarev, and further studied by Dlab-Ringel. Again by abuse of terminology, a point  $B$  in  $\mu^{-1}(0)$  is often called a representation of the preprojective algebra associated with  $Q^{\text{dbl}}$  (or  $Q$ ).

Let us explain another related interpretation of  $\mu$ . Let us take a point  $B \in \mathbf{M}(V)$  and consider

$$(2.1) \quad \begin{aligned} \bigoplus_{i \in Q_0} \text{End}(V_i) \xrightarrow{\iota} \mathbf{M}(V) \xrightarrow{d\mu} \bigoplus_{i \in Q_0} \text{End}(V_i), \\ \iota(\xi) = (\xi_{i(h)} B_h - B_h \xi_{o(h)})_h, \quad d\mu_i(C) = \sum_{\substack{h \in Q_1^{\text{dbl}} \\ i(h)=i}} \varepsilon(h) (B_h C_{\bar{h}} + C_h B_{\bar{h}}). \end{aligned}$$

The linear map  $\iota$  is nothing but the differential of the  $G_V$ -action given by conjugation, when we understand  $\bigoplus \text{End}(V_i)$  as the Lie algebra of  $G_V$ . On the other hand,  $d\mu$  is the differential of the moment map  $\mu$ . Note also that this is a complex, i.e.,  $d\mu \circ \iota = 0$  if  $\mu(B) = 0$ .

Now we observe

(2.2)  $d\mu$  is the transpose of  $\iota$  when we identify  $\mathbf{M}(V)$  with its dual space via the symplectic form.

( $\text{End}(V_i)$  is self-dual by the trace pairing.) Hence we have

$$(2.3) \quad \text{Ker } \iota \cong (\text{Cok } d\mu)^\vee, \quad \text{Ker } d\mu \cong (\text{Cok } \iota)^\vee.$$

As we mentioned above, we will consider a modification of a quotient space  $\mu^{-1}(0)/G_V$  later, which is a smooth algebraic variety. Let us omit the detail at this stage, and assume  $\mu^{-1}(0)/G_V$  is smooth so that the quotient map  $\mu^{-1}(0) \rightarrow \mu^{-1}(0)/G_V$  is a submersion. Then the tangent space of  $\mu^{-1}(0)/G_V$  at  $[B]$  is given by

$$T_{[B]}(\mu^{-1}(0)/G_V) = \text{Ker } d\mu / \text{Im } \iota.$$

Here  $[B]$  denotes the point in  $\mu^{-1}(0)/G_V$  given by  $B \in \mu^{-1}(0)$ . From the observation (2.3) above, the right hand side has the induced symplectic form. Let us denote it  $\omega$ . We consider  $\omega$  as a differential form on the manifold  $\mu^{-1}(0)/G_V$ . Let us check that  $\omega$  is closed, i.e.,  $d\omega = 0$ . In fact, it is enough to check that the pull-back of  $\omega$  to  $\mu^{-1}(0)$  is closed as the quotient map is a submersion. By the definition, the pull-back is nothing but the restriction of the symplectic form on  $\mathbf{M}(V)$ . Then as  $d$  commutes with the restriction, the closedness of the pull-back follows from that of the symplectic form on  $\mathbf{M}(V)$ . But the latter is trivial as  $\mathbf{M}(V)$  is a vector space and its symplectic form is constant.

Let us note that the complex (2.1) can be modified to one associated with a pair  $B^1 \in \mathbf{M}(V^1)$ ,  $B^2 \in \mathbf{M}(V^2)$  where both satisfy  $\mu = 0$ :

$$(2.4) \quad \bigoplus_{i \in Q_0} \text{Hom}(V_i^1, V_i^2) \xrightarrow{\alpha} \bigoplus_{h \in Q_1^{\text{dbl}}} \text{Hom}(V_{o(h)}^1, V_{i(h)}^2) \xrightarrow{\beta} \bigoplus_{i \in Q_0} \text{Hom}(V_i^1, V_i^2)$$

$$\alpha(\xi) = (\xi_{i(h)} B_h^1 - B_h^2 \xi_{o(h)})_h,$$

$$\beta(C, D, E) = \sum_{\substack{h \in Q_1^{\text{dbl}} \\ i(h)=i}} \varepsilon(h) (B_h^2 C_h + C_h B_h^1).$$

This complex is important in the representation theory of preprojective algebras. Let us regard  $B^1, B^2$  as modules of the preprojective algebra  $\Pi(Q)$ . Then we have

$$\text{Ker } \alpha \cong \text{Hom}_{\Pi(Q)}(B^1, B^2), \quad \text{Coker } \beta \cong \text{Hom}_{\Pi(Q)}(B^2, B^1)^\vee,$$

$$\text{Ker } \beta / \text{Im } \alpha \cong \text{Ext}_{\Pi(Q)}^1(B^1, B^2).$$

The first two isomorphisms are just by definition and the computation of the transpose of  $\beta$  as above. The last isomorphism is proved in [CB00].

From this observation, the quotient space  $\mu^{-1}(0)/G_V$  is a *nonlinear* version of the self-extension  $\text{Ext}_{\Pi(Q)}^1(B, B)$ , as a tangent space is linear approximation of a manifold. This partly explains importance of study of  $\mu^{-1}(0)/G_V$ , as  $\text{Ext}^1$  is a fundamental object in representation theory. It is also *deeper* than  $\text{Ext}^1$ , as the tangent space only reflects a local structure of the manifold, and cannot see global structures, such as topology of the manifold.

**2.2. Framed representations.** Now we take an additional  $Q_0$ -graded finite-dimensional complex vector space  $W = \bigoplus_{i \in Q_0} W_i$  and introduce

$$\mathbf{M}(V, W) = \bigoplus_{h \in Q_1^{\text{dbl}}} \text{Hom}(V_{o(h)}, V_{i(h)}) \oplus \bigoplus_{i \in Q_0} \text{Hom}(W_i, V_i) \oplus \text{Hom}(V_i, W_i).$$

An element of the additional factor  $\bigoplus_{i \in Q_0} \text{Hom}(W_i, V_i) \oplus \text{Hom}(V_i, W_i)$  is called a *framing* of a quiver representation, and we denote it by  $I = (I_i)_{i \in Q_0}$ ,  $J = (J_i)_{i \in Q_0}$ . A point  $(B, I, J) \in \mathbf{M}(V, W)$  is called a *framed representation*.

We have an action of  $G_V$ , and also of  $G_W = \prod_{i \in Q_0} \text{GL}(W_i)$ , on  $\mathbf{M}(V, W)$  by conjugation. It will be important for various applications of quiver varieties, but we will *not* use the latter action in this review.

We have the moment map

$$\mu = (\mu_i)_i: \mathbf{M}(V, W) \rightarrow \bigoplus_i \mathfrak{gl}(V_i); \quad \mu_i(B, I, J) = \sum_{\substack{h \in Q_1^{\text{dbl}} \\ i(h)=i}} \varepsilon(h) B_h B_{\bar{h}} + I_i J_i,$$

as above.

*Remark 1.* The framing factor naturally appeared in [KN90], and it is also important in applications of quiver varieties to representation theory of Lie algebras, as  $\dim W$  will be identified with a highest weight of a representation. However the author could not find earlier appearances in quiver representation literature, and he gave an explanation for the ring and representation theory community in [Na96].

On the other hand, Crawley-Boevey [CB01] found the following trick, which makes  $\mathbf{M}(V, W)$  as the previous  $\mathbf{M}(V')$  with a different quiver and a graded vector space  $V'$  [CB01]: Add a vertex  $\infty$  to  $Q$ , and draw edges from  $\infty$  to  $i \in Q_0$  as many as  $\dim W_i$ . We then defined  $V'$  as  $V$  plus one-dimensional vector space at the vertex  $\infty$ . We identify  $\text{Hom}(W_i, V_i)$  with  $\text{Hom}(\mathbb{C}, V_i)^{\oplus \dim W_i}$  after taking a base of  $W_i$ . Hence we have  $\mathbf{M}(V, W) = \mathbf{M}(V')$ .

*Remark 2.* In [Kr89, KN90] a deformation of the equation  $\mu = 0$  as  $\mu_i(B, I, J) = \zeta_i^{\mathbb{C}} \text{id}_{V_i}$  is considered. Here  $\zeta^{\mathbb{C}} = (\zeta_i^{\mathbb{C}})_i \in \mathbb{C}^{Q_0}$ . It motivated Crawley-Boevey and Holland [CH97] to study *deformed preprojective algebras*. We restrict our interest only on the undeformed case  $\zeta^{\mathbb{C}} = 0$  in this review, though many results remain true for deformed case.

Observation on symplectic forms for the case no framing  $W$  remains valid in the case with  $W$ , as  $\mathbf{M}(V, W)$  is still a symplectic vector space. In particular, we have an induced symplectic form on the quotient  $\mu^{-1}(0)/G_V$ .

Let us write down a framed analog of (2.4) for a pair  $(B^1, I^1, J^1) \in \mathbf{M}(V^1, W^1)$ ,  $(B^2, I^2, J^2) \in \mathbf{M}(V^2, W^2)$  where both satisfy  $\mu = 0$ :

$$\begin{aligned}
(2.5) \quad & \bigoplus_{i \in Q_0} \text{Hom}(V_i^1, V_i^2) \xrightarrow{\alpha} \bigoplus_{h \in Q_1^{\text{dbl}}} \text{Hom}(V_{o(h)}^1, V_{i(h)}^2) \oplus \bigoplus_{i \in Q_0} \text{Hom}(V_i^1, V_i^2) \xrightarrow{\beta} \bigoplus_{i \in Q_0} \text{Hom}(V_i^1, V_i^2) \\
& \bigoplus_{i \in Q_0} \text{Hom}(W_i^1, V_i^2) \oplus \text{Hom}(V_i^1, W_i^2) \\
& \alpha(\xi) = (\xi_{i(h)} B_h^1 - B_h^2 \xi_{o(h)})_h \oplus (\xi_i I_i^1)_i \oplus (-J_i^2 \xi_i)_i, \\
& \beta(C, D, E) = \sum_{\substack{h \in Q_1^{\text{dbl}} \\ i(h)=i}} \varepsilon(h) (B_h^2 C_h^1 + C_h B_h^1) + I_i^2 E_i + D_i J_i^1.
\end{aligned}$$

This complex appears at various points in study of quiver varieties, such as

- (1) the construction of instantons on an ALE space [KN90, (4.3)],
- (2) the tautological homomorphism in the definition of Kashiwara crystal structure on the set of irreducible components of lagrangian subvarieties [Na98, §4],
- (3) the definition of the Hecke correspondence [Na98, §5],
- (4) the decomposition of the diagonal [Na98, §6],
- (5) the definition of tensor product varieties [Na01, §3].

### 3. GIT QUOTIENTS

Since the group  $G_V$  is noncompact, the quotient topology on  $\mu^{-1}(0)/G_V$  is *not* Hausdorff in general. The trouble is caused by nonclosed  $G_V$ -orbits: If orbits  $O_1, O_2$  intersect in their closure  $\overline{O_1} \cap \overline{O_2}$ , the corresponding points in  $\mu^{-1}(0)/G_V$  cannot be separated by disjoint open neighborhoods.

**3.1. Affine quotients.** One solution to this problem is to introduce a coarser equivalence relation

$$x \sim y \iff \overline{G_V x} \cap \overline{G_V y} \neq \emptyset.$$

Then the quotient space  $\mu^{-1}(0)/\sim$  is a Hausdorff space. Let us denote this space by  $\mu^{-1}(0)//G_V$ . It is known that it has a structure of an affine algebraic scheme, in fact we have

$$\mu^{-1}(0)//G_V = \text{Spec } \mathbb{C}[\mu^{-1}(0)]^{G_V},$$

where  $\mathbb{C}[\mu^{-1}(0)]$  is the coordinate ring of the affine scheme  $\mu^{-1}(0)$ , and  $\mathbb{C}[\mu^{-1}(0)]^{G_V}$  is its  $G_V$ -invariant part. It is a fundamental theorem (due to Nagata) in geometric invariant theory that the invariant ring is finitely generated. The space  $\mu^{-1}(0)//G_V$  is called the *affine algebro-geometric quotient* of  $\mu^{-1}(0)$  by  $G_V$ . Let us denote it by  $\mathfrak{M}_0(V, W)$ .

The ring of invariants is generated by two types of functions: (1) Take an oriented cycle in the doubled quiver  $Q^{\text{dbl}}$  and consider the trace of the composition of corresponding linear maps. (2) Take a path starting from  $i$  to  $j$  and consider the composition of  $I_i$ , linear maps for edges in the path, and  $J_j$ , a linear map  $W_i \rightarrow W_j$ . Then its entry is an invariant function. This follows from [LP90] after Crawley-Boevey's trick.

When  $W = 0$ ,  $\mathfrak{M}_0(V, 0)$  parametrizes semisimple representation of the preprojective algebra  $\Pi(Q)$ . Roughly it is proved as follows. Suppose a representation  $B$  has a sub-representation  $B'$ . We have a short exact sequence  $0 \rightarrow B' \rightarrow B \rightarrow B/B' \rightarrow 0$ . By the action of ‘triangular’ elements in  $G_V$ , we can send the off-diagonal entries to 0, in other words,  $B$  can be degenerated to the direct sum  $B' \oplus (B/B')$ . But  $\mathfrak{M}_0(V, 0)$  parametrizes closed orbits, hence  $B \cong B' \oplus (B/B')$ . It means that we can take complementary sub-representation of  $B'$ . We continue this process until it becomes a direct sum of simple representations.

The same is true even for  $W \neq 0$  if we understand *semisimple* representations appropriately.

We can consider similar quotients of  $\mathbf{M}(V)$  or  $\mathbf{N}(V)$ . But they are often simple spaces:

**Example 3.** (1) Consider a quiver  $Q$  without oriented cycles (e.g., a finite *ADE* quiver) and define the affine algebro-geometric quotient  $\mathbf{N}(V)//G_V$  as above. Since we do not have oriented cycles, there is no invariant function. Hence  $\mathbf{N}(V)//G_V$  consists of a single point  $\{0\}$ .

(2) Let  $V$  be an  $n$ -dimensional complex vector space. Consider a  $\mathrm{GL}(V)$ -action on  $\mathrm{End}(V)$  given by conjugation. Then  $\mathrm{End}(V)//\mathrm{GL}(V)$  is identified with  $\mathbb{C}^n/\mathfrak{S}_n$ , the space of eigenvalues up to permutation. Here  $\mathfrak{S}_n$  is the symmetric group of  $n$  letters.

With a little more effort, one show

*Exercise 4.* (1) Consider an *ADE* quiver  $Q$  and define the affine algebro-geometric quotient  $\mathfrak{M}_0(V, 0) = \mu^{-1}(0)//G_V$  for  $W = 0$ . Show that it consists of a single point  $\{0\}$ . (This can be deduced from Lusztig’s result saying that  $B \in \mu^{-1}(0)$  is always nilpotent for an *ADE* quiver. Alternative proof is given in [Na94, Prop. 6.7].)

(2) Consider the Jordan quiver with an  $n$ -dimensional vector space  $V$  and  $W = 0$ . Then  $\mathfrak{M}_0(V, 0) = \mu^{-1}(0)//\mathrm{GL}(V)$  is  $(\mathbb{C}^2)^n/\mathfrak{S}_n$ , the space of pairs of eigenvalues of  $B_1, B_2$  up to permutation.

On the other hand  $\mathfrak{M}_0(V, W)$  (in general) and  $\mathfrak{M}_0(V, 0)$  for non *ADE* quiver are quite often complicated spaces.

**Example 5.** Consider the  $A_1$  quiver with vector spaces  $V, W$ . Then  $\mathbf{M}(V, W) = \mathrm{Hom}(W, V) \oplus \mathrm{Hom}(V, W)$  with  $\mu(I, J) = IJ$ . We consider  $A = JI \in \mathrm{End}(W)$ . It is invariant under  $\mathrm{GL}(V)$  and its entries are  $\mathrm{GL}(V)$ -invariant functions on  $\mathbf{M}(V, W)$ . A fundamental theorem of the invariant theory says that they generate the ring of invariants. It satisfies  $A^2 = JIJI = 0$  if  $\mu(I, J) = 0$ . A little more effort shows

$$\mathfrak{M}_0(V, W) = \{A \in \mathrm{End}(W) \mid A^2 = 0, \mathrm{rank} A \leq \dim V\}.$$

Note that  $\mathbf{N}(V, W)//\mathrm{GL}(V)$  is  $\{0\}$  in this example.

**3.2. GIT quotients.** Another way to construct a nice quotient space is to take a  $G_V$ -invariant open subset  $U$  of  $\mu^{-1}(0)$  so that arbitrary  $G_V$ -orbit in  $U$  is closed (in  $U$ ). Such an open subset  $U$  arises in geometric invariant theory. Since it is not our intension to explain detailed structures of the quotient as an algebraic variety, let us directly goes to a definition of the open subset  $U$ . In fact, it depends on a choice, the stability parameter  $\zeta^{\mathbb{R}} = (\zeta_i^{\mathbb{R}}) \in \mathbb{R}^{Q_0}$ .

We consider  $\zeta^{\mathbb{R}}$  as a function  $\mathbb{Z}^{Q_0} \rightarrow \mathbb{R}$  by  $\zeta^{\mathbb{R}}((v_i)_i) = \sum \zeta_i^{\mathbb{R}} v_i$ .

**Definition 6.** We say  $(B, I, J) \in \mathbf{M}(V, W)$  is  $\zeta^{\mathbb{R}}$ -semistable if the following two conditions are satisfied:

- (1) If a  $Q_0$ -graded subspace  $S = \bigoplus S_i$  in  $V$  is contained in  $\text{Ker } J$  and  $B$ -invariant, then  $\zeta^{\mathbb{R}}(\dim S) \leq 0$ .
- (2) If a  $Q_0$ -graded subspace  $T = \bigoplus T_i$  in  $V$  contains  $\text{Im } I$  and  $B$ -invariant, then  $\zeta^{\mathbb{R}}(\dim T) \leq \zeta^{\mathbb{R}}(\dim V)$ .

We say  $(B, I, J)$  is  $\zeta^{\mathbb{R}}$ -stable if the strict inequalities hold in (1),(2) unless  $S = 0, T = V$  respectively.

*Remark 7.* In view of Remark 1, we can express the condition in terms of  $\mathbf{M}(V')$ : we set  $\zeta_{\infty}^{\mathbb{R}} = -\zeta^{\mathbb{R}}(\dim V)$ . Then  $\zeta^{\mathbb{R}}$  is extended to a function  $\mathbb{Z}^{Q_0 \sqcup \{\infty\}} \rightarrow \mathbb{R}$ . Then the condition is equivalent to  $\zeta^{\mathbb{R}}(\dim S') \leq 0$  for a graded invariant subspace  $S' \subset V'$ . According to either  $S'_{\infty} = 0$  or  $\mathbb{C}$ , we have the above two cases (1), (2) respectively. Note that  $S'$  being invariant means  $S'$  is a submodule. Hence this reformulation coincides with the standard King's stability condition [Ki94].

Let us give a simple consequence of the  $\zeta^{\mathbb{R}}$ -stability condition. It basically says a  $\zeta^{\mathbb{R}}$ -stable framed representation is Schur:

**Proposition 8.** *Suppose  $(B, I, J)$  is  $\zeta^{\mathbb{R}}$ -stable. Then the kernel of  $\iota$  and cokernel of  $d\mu$  in  $q$  (the framed version of) (2.1) are trivial.*

*Proof.* By (2.2) it is enough to check the assertion for  $\iota$ . Suppose  $\xi = (\xi_i)_i$  is in  $\text{Ker } \iota$ . Then  $\text{Im } \xi$  is  $B$ -invariant and contained in  $\text{Ker } J$ . Therefore  $\zeta^{\mathbb{R}}(\dim \text{Im } \xi) \leq 0$  by the  $\zeta^{\mathbb{R}}$ -semistability condition. Similarly  $\text{Ker } \xi$  is  $B$ -invariant and contains  $\text{Im } I$ . Therefore  $\zeta^{\mathbb{R}}(\dim \text{Ker } \xi) \leq \zeta^{\mathbb{R}}(\dim V)$ . But  $\zeta^{\mathbb{R}}(\dim \text{Im } \xi) + \zeta^{\mathbb{R}}(\dim \text{Ker } \xi) = \zeta^{\mathbb{R}}(\dim V)$ . Therefore two inequalities must be equalities. The  $\zeta^{\mathbb{R}}$ -stability condition says  $\text{Im } \xi = 0$  and  $\text{Ker } \xi = V$ . These are nothing but  $\xi = 0$ .  $\square$

This also implies that the stabilizer of a stable point  $(B, I, J)$  in  $G_V$  is trivial: If  $g = (g_i)_i$  stabilizes  $(B, I, J)$ ,  $(g_i - \text{id}_{V_i})_i$  is in the kernel of  $\iota$ , hence must be trivial by the proposition.

The  $G_V$ -orbit through  $(B, I, J)$  is of the form  $G_V / \text{Stabilizer}$ . Hence all  $\zeta^{\mathbb{R}}$ -stable orbits have the maximal dimension, equal to  $\dim G_V$ . Since an orbit  $O_1$  appeared in the closure of an orbit  $O_2$  has  $\dim O_1 < \dim O_2$ , we conclude that  $\zeta^{\mathbb{R}}$ -orbits are closed in the open subset of all  $\zeta^{\mathbb{R}}$ -stable points in  $\mathbf{M}(V, W)$ .

Let  $\mu^{-1}(0)_{\zeta^{\mathbb{R}}}^s$  be the subset of  $\zeta^{\mathbb{R}}$ -stable points in  $\mu^{-1}(0)$ .

**Theorem 9.** (1)  $\mu^{-1}(0)_{\zeta^{\mathbb{R}}}^s$  is a complex manifold (i.e., nonsingular) whose dimension is  $\dim \mathbf{M}(V, W) - \dim G_V$ .

(2) The quotient  $\mu^{-1}(0)_{\zeta^{\mathbb{R}}}^s / G_V$  is a complex manifold whose dimension is  $\dim \mathbf{M}(V, W) - 2 \dim G_V$ .

The first assertion is a simple consequence of the inverse function theorem, as  $d\mu$  is surjective over  $\mu^{-1}(0)_{\zeta^{\mathbb{R}}}^s$ . The second assertion is a little more difficult to prove, but it is a consequence of the assertion that the  $G_V$ -action on  $\mu^{-1}(0)_{\zeta^{\mathbb{R}}}^s$  is free and closed. For our applications, this will be important as Poincaré duality isomorphism holds for  $\mu^{-1}(0)_{\zeta^{\mathbb{R}}}^s / G_V$ .



It is known that the  $\zeta^{\mathbb{R}}$ -semistability automatically implies the  $\zeta^{\mathbb{R}}$ -stability unless  $\zeta^{\mathbb{R}}$  lies in a finite union of hyperplanes in  $\mathbb{R}^{Q_0}$ . In this case, it is known that the natural map

$$(3.1) \quad \pi: \mu^{-1}(0)_{\zeta^{\mathbb{R}}}^s / G_V \rightarrow \mu^{-1}(0) // G_V$$

is proper, i.e., inverse images of compact subsets remain compact. Here the map is defined by assigning  $\sim$ -equivalence classes to  $\zeta^{\mathbb{R}}$ -stable  $G_V$ -orbits. If  $(B, I, J)$  is regarded as a framed representation of the preprojective algebra, it is sent to its ‘semisimplification’ under  $\pi$ .

When  $\zeta^{\mathbb{R}}$  lies in a finite union of hyperplanes,  $\pi$  is not proper. We need to replace  $\mu^{-1}(0)_{\zeta^{\mathbb{R}}}^s / G_V$  by a larger space, a certain quotient of the space of  $\zeta^{\mathbb{R}}$ -semistable points in  $\mu^{-1}(0)$ , similar to  $\mu^{-1}(0) // G_V$ . We do not consider such  $\zeta^{\mathbb{R}}$ . We always assume  $\zeta^{\mathbb{R}}$ -stability and  $\zeta^{\mathbb{R}}$ -semistability are equivalent hereafter.

Let us denote  $\mu^{-1}(0)_{\zeta^{\mathbb{R}}}^s / G_V$  by  $\mathfrak{M}_{\zeta^{\mathbb{R}}}(V, W)$ . We will use the case  $\zeta_i^{\mathbb{R}} > 0$  for all  $i \in Q_0$  later. In this case we simply denote it by  $\mathfrak{M}(V, W)$ . The inverse image  $\pi^{-1}(0)$  will be important. Let us denote it by  $\mathfrak{L}(V, W)$ .

For a  $\zeta^{\mathbb{R}}$ -stable framed representation  $(B, I, J)$ , the corresponding point in  $\mathfrak{M}_{\zeta^{\mathbb{R}}}(V, W)$  is denoted by  $[B, I, J]$ .

**Example 10.** Consider the  $A_1$  quiver and vector spaces  $V, W$  as in Example 5. When the stability parameter  $\zeta^{\mathbb{R}} > 0$  (resp.  $\zeta^{\mathbb{R}} < 0$ ), the  $\zeta^{\mathbb{R}}$ -semistability means that  $J$  is injective (resp.  $I$  is surjective). Note also that  $\zeta^{\mathbb{R}}$ -semistability and  $\zeta^{\mathbb{R}}$ -stability are equivalent. Suppose  $\zeta^{\mathbb{R}} > 0$  for brevity. Then  $\text{Im } J$  is a subspace of  $W$  with dimension  $\dim V$ . Hence we have a map  $\mathfrak{M}(V, W) \rightarrow \text{Gr}(V, W)$ , the Grassmannian variety of subspaces in  $W$  with dimension  $\dim V$ . In particular, we have  $\mathfrak{M}(V, W) = \emptyset$  unless  $0 \leq \dim V \leq \dim W$ .

Consider  $A = JI$  as in Example 5. We have  $\text{Im } A \subset \text{Im } J$  and  $\text{Im } A \subset \text{Ker } A$ , hence  $A \in \text{Hom}(W / \text{Im } J, \text{Im } J)$ . Moreover it is simple to check that  $A$  together with  $\text{Im } J$  conversely determines  $(I, J)$  up to  $\text{GL}(V)$ -action. This shows that  $\mathfrak{M}(V, W) \cong T^* \text{Gr}(V, W)$ , the cotangent bundle of  $\text{Gr}(V, W)$ .

The map  $\pi$  in (3.1) is given by  $(\text{Im } J, A) \mapsto A$ . Comparing with Example 5, one see that  $\pi$  is surjective when  $\dim V \leq \dim W / 2$ . In fact, it is known that it is a resolution of singularities. On the other hand, the image is a proper subset if  $\dim V > \dim W / 2$  as  $\text{rank } A \leq \dim W - \dim V < \dim W / 2$ . Note also that  $\mathfrak{L}(V, W) = \text{Gr}(V, W)$ .

In this example  $\mathfrak{M}(V, W)$  is a cotangent bundle of  $\text{Gr}(V, W)$  which is the quotient of  $\zeta^{\mathbb{R}}$ -semistable points in  $\mathbf{N}(V, W)^* = \text{Hom}(V, W)$  by  $\text{GL}(V)$ . But it is not the case as the following example illustrate:

**Example 11.** Consider the quiver of type  $A_n$  with  $\dim V_i = 1$  for all  $i \in Q_0$ ,  $\dim W_i = 1$  for  $i = 1, n$ ,  $\dim W_i = 0$  for  $i \neq 1, n$ . Here we number vertices as usual.

$$\begin{array}{ccccccc}
 \mathbb{C} & \xleftarrow{B_{1,2}} & \mathbb{C} & \xleftarrow{B_{2,3}} & \mathbb{C} & \xleftarrow{B_{3,4}} & \cdots & \xleftarrow{B_{n-2,n-1}} & \mathbb{C} & \xleftarrow{B_{n-1,n}} & \mathbb{C} \\
 I_1 \updownarrow J_1 & & & & & & & & & & I_n \updownarrow J_n \\
 \mathbb{C} & & & & & & & & & & \mathbb{C}
 \end{array}$$

The ring of invariant functions is generated by

$$x = J_n B_{n,n-1} \cdots B_{2,1} I_1, \quad y = J_1 B_{1,2} \cdots B_{n-1,n} I_n, \quad z = J_1 I_1,$$

which satisfies  $xy = z^{n+1}$  thanks to equations  $I_1 J_1 = B_{1,2} B_{2,1}$ , etc (up to sign). Thus  $\mathfrak{M}_0(V, W)$  is the hypersurface  $xy = z^{n+1}$  in  $\mathbb{C}^3$ .

Now we take the stability parameter  $\zeta^{\mathbb{R}}$  with  $\zeta_i^{\mathbb{R}} > 0$  for all  $i$ . Let us study  $\pi: \mathfrak{M}(V, W) \rightarrow \mathfrak{M}_0(V, W)$ . One first check that  $(B, I, J)$  is  $\zeta^{\mathbb{R}}$ -stable if it corresponds to  $(x, y, z) \neq (0, 0, 0)$ . In fact, there are no subspaces  $S, T$  appearing in the definition of  $\zeta^{\mathbb{R}}$ -stability in this case. An interesting thing happens when  $(x, y, z) = (0, 0, 0)$ . Starting from  $J_1 I_1 = 0$ , we have  $B_{i,i+1} B_{i+1,i} = 0$   $i = 1, \dots, n-1$ , and  $I_n J_n = 0$  thanks to  $\mu = 0$ . Since all vector spaces have dimension 1, at least one of paired linear maps is zero. On the other hand, the  $\zeta^{\mathbb{R}}$ -stability condition means that it is not possible that two linear maps starting from  $V_i$  ( $1 \leq i \leq n$ ) cannot be simultaneously zero, as  $V_i$  violates the condition then. Then one check that the only possibility is

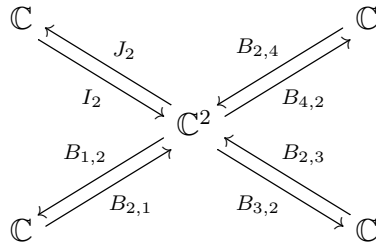
$$\begin{array}{ccccccccccc} \mathbb{C} & \xleftarrow{B_{1,2}} & \mathbb{C} & \xleftarrow{B_{2,3}} & \dots & \xleftarrow{B_{i-1,i}} & \mathbb{C} & \xrightarrow{B_{i+1,i}} & \dots & \xrightarrow{B_{n-1,n-2}} & \mathbb{C} & \xrightarrow{B_{n,n-1}} & \mathbb{C} \\ & \downarrow J_1 & & & & & & & & & & & \downarrow J_n \\ & \mathbb{C} & & & & & & & & & & & \mathbb{C} \end{array}$$

for some  $i = 1, \dots, n$ . Here only nonzero maps are written. By the  $G_V$ -action, we can normalize all maps as 1 except  $B_{i+1,i}, B_{i-1,i}$ . Then the remaining data is  $(B_{i+1,i}, B_{i-1,i}) \in \mathbb{C}^2 \setminus 0$  modulo the action of  $\mathrm{GL}(V_i) = \mathbb{C}^\times$ . We thus get the complex projective line  $\mathbb{C}P^1$ . Let us denote this by  $\mathfrak{C}_i$ . Thus we have  $\mathfrak{L}(V, W) = \mathfrak{C}_1 \cup \mathfrak{C}_2 \cup \dots \cup \mathfrak{C}_n$ . The intersection  $\mathfrak{C}_i \cap \mathfrak{C}_{i+1}$  is  $B_{i+1,i} = 0$ , hence is a single point. Other intersection  $\mathfrak{C}_i \cap \mathfrak{C}_j$  is empty. Thus  $\mathfrak{C}_i$ 's form a chain of  $n$  projective lines. In fact,  $\mathfrak{M}(V, W)$  is the minimal resolution of the simple singularity  $xy = z^n$  of type  $A_n$ .

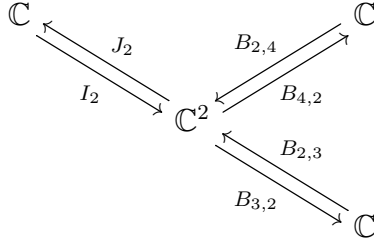
This example can be generalized to other  $ADE$  singularities as follows. Take an affine Dynkin diagram of type  $ADE$  and consider the primitive (positive) imaginary root vector  $\delta$ . We remove the special vertex 0 and take the corresponding vector space  $V$  of the finite  $ADE$  quiver. The entry of the special vertex 0 is always 1, and let us make it to  $W$  for the finite  $ADE$  quiver. For type  $A_n$ , the vertex 0 is connected to 1 and  $n$ , hence we set  $W_1 = \mathbb{C}$ ,  $W_n = \mathbb{C}$  and  $W_i = 0$  otherwise as the above example. For other types, 0 is connected to a single vertex, say  $i_0$ . Hence we take  $W_{i_0} = \mathbb{C}$  and  $W_i = 0$  otherwise. Then  $\mathfrak{M}_0(V, W)$  is the simple singularity of the corresponding type, and  $\mathfrak{M}(V, W)$  is its minimal resolution. This is nothing but Kronheimer's construction [Kr89].

The exceptional set of the minimal resolution, i.e., the inverse image of 0 under  $\pi$  (which is our  $\mathfrak{L}(V, W)$ ) is known to be union of projective lines intersecting as the Dynkin diagram. Let us check this assertion for  $D_4$ .

**Example 12.** We consider  $\mathfrak{M}(V, W)$ ,  $\mathfrak{M}_0(V, W)$  of type  $D_4$  with

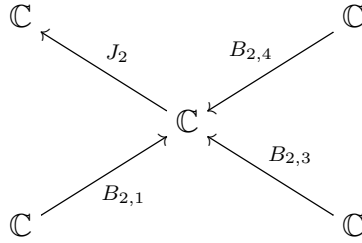


where the upper left vector space is  $W_2$  and others are  $V_i$ 's. As is observed in Example 11, it is helpful to consider a vector subspace where there are no coming linear maps. Suppose  $V_1$  (the left lower space) is so, i.e.,  $B_{1,2} = 0$ . Then the data with  $V_1$  removed, i.e.,



is also  $\zeta$ -stable. It is easy to check that the corresponding space  $\mathfrak{M}(V', W)$  ( $V' = V \ominus V_1$ ) is a single point. Conversely we start from the point  $\mathfrak{M}(V', W)$  and add  $B_{2,1}$  to get a point in  $\mathfrak{M}(V, W)$ . As in the type  $A_n$  case, points constructed in this way form the complex line. Let us denote it by  $\mathfrak{C}_1$ . Replacing  $V_1$  by  $V_3, V_4$ , we have  $\mathfrak{C}_3, \mathfrak{C}_4$ .

Let us focus on  $V_2$ . Contrary to other vertices, we cannot remove the whole  $V_2$ , as it violates the  $\zeta^{\mathbb{R}}$ -stability. We instead replace  $V_2$  by one dimensional space  $V_2$ . Then all vector spaces are 1-dimensional, and it is easy to check that the corresponding variety  $\mathfrak{M}(V', W)$  is a single point given by



All written maps are nonzero. When we add one dimensional vector space to  $V_2'$ , we consider it as a subspace in  $V_1 \oplus V_3 \oplus V_4$  by  $B_{1,2}, B_{3,2}, B_{4,2}$ . Since  $\mu = 0$  is satisfied, it must be contained in the kernel of  $(B_{2,1}, B_{2,3}, B_{2,4}): V_1 \oplus V_3 \oplus V_4 \rightarrow V_2'$ , which is a 2-dimensional space. Therefore points constructed in this way also form the complex line, denoted by  $\mathfrak{C}_2$ . It is also clear that  $\mathfrak{C}_2$  meets with  $\mathfrak{C}_1, \mathfrak{C}_3, \mathfrak{C}_4$  at three distinct points, hence the configuration forms the Dynkin diagram  $D_4$ .

Subvarieties  $\mathfrak{C}_i$  are examples of Hecke correspondence, defined in §4.4, where the factor  $\mathfrak{M}(V', W)$  is a single point as we have seen above, hence is a subvariety in  $\mathfrak{M}(V, W)$ . It will be also clear that why  $\mathfrak{C}_i$  is a projective space : it is a projective space associated with a certain  $\text{Ext}^1$ .

#### 4. REPRESENTATIONS OF KAC-MOODY LIE ALGEBRAS

In this section we assume that the quiver  $Q$  has no edge loops. Therefore we have the (symmetric) Kac-Moody Lie algebra  $\mathfrak{g} = \mathfrak{g}_Q$  whose Dynkin diagram is the graph obtained from  $Q$  by replacing oriented arrows by unoriented edges. If  $Q$  is of type  $ADE$ , the Kac-Moody Lie algebra is a complex simple Lie algebra of the corresponding type.

*Remark 13.* When  $Q$  has an edge loop, say the Jordan quiver, it was not a priori clear what is an analog of  $\mathfrak{g}_Q$ . Recently Maulik-Okounkov find a definition of a Lie algebra based on

quiver varieties possibly with edge loops [MO12]. Bozec also studies a generalized crystal structure on the set of irreducible components [Bo16].

#### 4.1. Lagrangian subvariety.

**Theorem 14** ([Na94, Th. 5.8]).  $\mathfrak{L}(V, W)$  is a lagrangian subvariety in  $\mathfrak{M}(V, W)$ . In particular, all irreducible components of  $\mathfrak{L}(V, W)$  has dimension  $\dim \mathfrak{M}(V, W)/2$ .

The proof is geometric, hence is omitted. We at least see that it is true for above examples. One can also check that it is not true for Jordan quiver with  $\dim V = \dim W = 1$ , as  $\mathfrak{M}(V, W) = \mathbb{C}^2$ ,  $\mathfrak{L}(V, W) = \{0\}$ . Thus it is important to assume that  $Q$  has no edge loops.

We consider the top degree homology group of  $\mathfrak{L}(V, W)$

$$H_{d(V,W)}(\mathfrak{L}(V, W)),$$

where  $d(V, W) = \dim_{\mathbb{C}} \mathfrak{M}(V, W) = \dim \mathbf{M}(V, W) - 2 \dim G_V$ . It should not be confused with cohomology groups of modules.  $\mathfrak{L}(V, W)$  is a topological space with classical topology, and we consider its singular homology group. Here we consider homology groups with *complex* coefficients, though we can consider *integer* coefficients also. It is known that  $\mathfrak{L}(V, W)$  is a lagrangian subvariety in  $\mathfrak{M}(V, W)$  with respect to the symplectic structure explained in the previous section. In particular, its dimension is half of  $d(V, W)$ , hence the above is the top degree homology. Thus  $H_{d(V,W)}(\mathfrak{L}(V, W))$  has a base given by irreducible components of  $\mathfrak{L}(V, W)$ .

A reader who is not comfortable with homology groups could use the space of constructible functions on  $\mathfrak{L}(V, W)$  instead. The definition of the action is in parallel, though the construction of a base corresponding to irreducible components is more involved. The construction of the base is due to Lusztig, and is called *semicanonical base*.

**4.2. Examples.** Our main goal in this section is to explain that the direct sum  $\bigoplus_V H_{d(V,W)}(\mathfrak{L}(V, W))$  has a structure of an integrable representation of  $\mathfrak{g}$  with highest weight  $\dim W$ . Let us first check it in the level of dimension (or weights).

Take  $A_1$  as in Example 10. We have

$$\dim H_{d(V,W)}(\mathfrak{L}(V, W)) = \begin{cases} 1 & \text{if } 0 \leq \dim V \leq \dim W, \\ 0 & \text{otherwise.} \end{cases}$$

This is the same as weight spaces of the finite dimensional irreducible representation of  $\mathfrak{g} = \mathfrak{sl}(2)$  with highest weight  $n = \dim W$ .

Since this is a review for the proceeding of Symposium on Ring Theory and Representation Theory, let us review the usual construction of this representation. It is realized as the space of degree  $n$  homogeneous polynomials in two variables:

$$\mathbb{C}x^n \oplus \mathbb{C}x^{n-1}y \oplus \cdots \oplus \mathbb{C}xy^{n-1} \oplus \mathbb{C}y^n.$$

Here the  $\mathfrak{sl}(2)$ -action is induced from that on  $\text{Span}(x, y) = \mathbb{C}^2$ . More concretely let us take a standard base of  $\mathfrak{sl}(2)$  as

$$(4.1) \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then

$$Hx = x, \quad Hy = -y, \quad Ex = 0, \quad Ey = x, \quad Fx = y, \quad Fy = 0.$$

The induced action means that  $H, E, F$  acts on homogeneous polynomials as derivation, for example

$$Hx^n = nx^{n-1}Hx = nx^n, \quad Ex^n = nx^{n-1}Ex = 0, \quad Fx^n = nx^{n-1}Fx = nx^{n-1}y, \quad \text{etc.}$$

Observe that  $x^n, x^{n-1}y, \dots, y^n$  are eigenvectors of  $H$  with eigenvalues  $n, n-2, \dots, -n$ . In this example, weight spaces are all 1-dimensional, and are scalar multiples of those vectors. (We have  $(n+1)$  eigenvectors in total, and the total dimension of the representation is  $(n+1)$ .)

Thus we see that dimension of weight spaces matches with dimension of homology groups above. At this stage it looks just a coincidence.

Next consider Example 11. From we saw there, we have

$$H_{d(V,W)}(\mathfrak{L}(V,W)) = H_2(\mathfrak{C}_1 \cup \mathfrak{C}_2 \cup \dots \cup \mathfrak{C}_n) = \mathbb{C}[\mathfrak{C}_1] \oplus \mathbb{C}[\mathfrak{C}_2] \oplus \dots \oplus \mathbb{C}[\mathfrak{C}_n],$$

where  $[\ ]$  denotes the fundamental class. In this example, the Lie algebra  $\mathfrak{g}$  is  $\mathfrak{sl}(n+1)$ , and the representation has the highest weight  $\dim W = \varpi_1 + \varpi_n$ , in other words, it is the adjoint representation, i.e., the Lie algebra itself considered as a representation with the action given by the Lie bracket. Since we choose a particular  $V$  (unlike  $A_1$  example above), the homology group corresponds to a weight space. In this example, we consider the zero weight space, which is the space of diagonal matrices in  $\mathfrak{sl}(n+1)$ . It is indeed  $n$ -dimensional.

Let us again spell out the weight spaces of the adjoint representation concretely.  $\mathfrak{sl}(n+1)$  is the space of trace-free  $(n+1) \times (n+1)$  complex matrices, regarded as a Lie algebra by the bracket  $[A, B] = AB - BA$ . Let us denote by  $\mathfrak{h}$  the space of diagonal matrices in  $\mathfrak{sl}(n+1)$ . It forms a commutative Lie subalgebra in  $\mathfrak{sl}(n+1)$ , and called a Cartan subalgebra. We have vector space decomposition

$$\mathfrak{sl}(n+1) = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathbb{C}E_{ij},$$

where  $E_{ij}$  is the matrix unit for the entry  $(i, j)$ . This is the simultaneous eigenspace decomposition of  $\mathfrak{sl}(n+1)$  with respect to the action of elements in  $\mathfrak{h}$ . The space  $\mathfrak{h}$  itself is the zero eigenspace, and  $E_{ij}$  is an eigenvector.

*Exercise 15.* Let  $W$  be the same as above, but consider  $\mathfrak{M}(V, W)$  for different  $V$ . Show that  $\mathfrak{M}(V, W)$  and  $\mathfrak{L}(V, W)$  are either empty set or a single point. Check that it coincides with the weight spaces of the adjoint representation of  $\mathfrak{sl}(n+1)$ . (Recall the homology group of the empty set is 0-dimensional vector space.) For example, if we remove  $\mathbb{C}$  at the  $i$ th vertex, it corresponds to  $\mathbb{C}E_{i, i+1}$ .

This is easy if all  $V_i$  are at most 1-dimensional (corresponding to the matrix unit  $E_{ij}$  with  $i < j$ ). But one needs to use the stability condition in an essential way if some  $V_i$  has dimension greater than 1.

One could also show that Example 12 corresponds to the adjoint representation of  $\mathfrak{g} = \mathfrak{so}(8)$  so that  $H_{d(V,W)}(\mathfrak{L}(V, W))$  is the space of diagonal matrices, and spaces for other  $V$  are either 0 or 1-dimensional. But this becomes even more tedious calculation and the

author never check it by mere analysis without using general structure theory explained below.

**4.3. Convolution product.** As we write above already, we use homology groups to give a geometric realization of representations of Kac-Moody Lie algebras. A reader who prefers constructible functions skip this subsection and goes to the next.

For homology groups, it is technically simpler to work with a version of the Borel-Moore homology group, which turns out to be isomorphic to the usual homology group for  $\mathfrak{L}(V, W)$ .<sup>1</sup> A review of the definition of the Borel-Moore homology group and its fundamental properties is found in [Fu96, App. B]. In our situation,  $\mathfrak{L}(V, W)$  is a closed subspace in a smooth oriented manifold  $\mathfrak{M}(V, W)$  of real dimension  $2d(V, W)$ . Then the Borel-Moore homology group is defined as

$$\begin{aligned} H_*(\mathfrak{L}(V, W)) &= H^{2d(V, W)-*}(\mathfrak{M}(V, W), \mathfrak{M}(V, W) \setminus \mathfrak{L}(V, W)) \\ &= H^{2d(V, W)-*}(\mathfrak{M}(V, W), \mathfrak{L}(V, W)^c). \end{aligned}$$

In fact, this definition makes sense for any embedding of  $\mathfrak{L}(V, W)$  into a smooth manifold, and is independent of the choice.

Let us take another  $Q_0$ -graded vector space  $V'$  and consider varieties  $\mathfrak{L}(V', W)$  also. Let us consider the fiber product  $Z(V, V', W)$ ,

$$Z(V, V', W) = \mathfrak{M}(V, W) \times_{\mathfrak{M}_0(V \oplus V', W)} \mathfrak{M}(V', W),$$

where  $\mathfrak{M}(V, W)$  (resp.  $\mathfrak{M}(V', W)$ )  $\rightarrow \mathfrak{M}_0(V \oplus V', W)$  is the composite of  $\pi: \mathfrak{M}(V, W)$  (resp.  $\mathfrak{M}(V', W)$ )  $\rightarrow \mathfrak{M}_0(V, W)$  (resp.  $\mathfrak{M}_0(V', W)$ ) and closed embeddings  $\mathfrak{M}_0(V, W)$  (resp.  $\mathfrak{M}(V', W)$ )  $\rightarrow \mathfrak{M}_0(V \oplus V', W)$  is given by setting data for  $B$  in  $V'$  (resp.  $V$ ) by 0. It is a closed subvariety in  $\mathfrak{M}(V, W) \times \mathfrak{M}(V', W)$ , and called *an analog of Steinberg variety* or a *Steinberg-type variety*, as a similar space is considered by Steinberg for the case of the cotangent bundle of a flag variety. Note that the restriction of projection  $p_1, p_2: Z(V, V', W) \rightarrow \mathfrak{M}(V, W), \mathfrak{M}(V', W)$  are proper (i.e., inverse images of compact subsets are compact) and  $p_2(p_1^{-1}(\mathfrak{L}(V, W))) \subset \mathfrak{L}(V', W)$ .

We define the Borel-Moore homology group of  $Z(V, V', W)$  as above, using  $\mathfrak{M}(V, W) \times \mathfrak{M}(V', W)$ . Suppose  $c \in H_k(Z(V, V', W))$ . Then we define the convolution product with  $c$  by

$$c * \alpha = p_{2*}(c \cap p_1^*(\alpha)), \quad \alpha \in H_{k'}(\mathfrak{L}(V, W)).$$

Let us check that this is well-defined step by step. First  $\alpha$  is in  $H^{2d(V, W)-k}(\mathfrak{M}(V, W), \mathfrak{L}(V, W)^c)$  as above. Then its pull-back  $p_1^*(\alpha)$  is  $H^{2d(V, W)-k}(\mathfrak{M}(V, W) \times \mathfrak{M}(V', W), p_1^{-1}(\mathfrak{L}(V, W))^c)$ . Its intersection  $c \cap p_1^*(\alpha)$  with  $c$  is a class in

$$H^{4d(V, W)+2d(V', W)-k-k'}(\mathfrak{M}(V, W) \times \mathfrak{M}(V', W), (p_1^{-1}(\mathfrak{L}(V, W)) \cap Z(V, V', W))^c),$$

as we consider  $c$  as an element of  $H^{2d(V, W)+2d(V', W)-k'}(\mathfrak{M}(V, W) \times \mathfrak{M}(V', W), Z(V, V', W)^c)$ . Hence  $c \cap p_1^*(\alpha)$  is a class in the Borel-Moore homology group  $H_{k+k'-2d(V, W)}(p_1^{-1}(\mathfrak{L}(V, W)) \cap Z(V, V', W))$ . By our assumption  $p_1^{-1}(\mathfrak{L}(V, W)) \cap Z(V, V', W)$  is a compact set, hence the pushforward homomorphism  $p_{2*}: H_*(p_1^{-1}(\mathfrak{L}(V, W)) \cap Z(V, V', W)) \rightarrow H_*(\mathfrak{L}(V', W))$  is defined. (See [Fu96, §B2].)

<sup>1</sup>This is because  $\mathfrak{L}(V, W)$  is a complex projective variety, hence a finite CW complex.

From the computation of degrees, if  $\alpha \in H_{d(V,W)}(\mathfrak{L}(V, W))$ , then  $c*\alpha \in H_{k-d(V,W)}(\mathfrak{L}(V', W))$ . Therefore if the degree  $k$  of  $c$  is  $d(V, W) + d(V', W)$ , the degree of  $c*\alpha$  is  $d(V', W)$ . Note that  $d(V, W) + d(V', W)$  is the complex dimension of  $\mathfrak{M}(V, W) \times \mathfrak{M}(V', W)$ , hence the degree of  $c$  is  $d(V, W) + d(V', W)$  means that it is a half-dimensional class in  $\mathfrak{M}(V, W) \times \mathfrak{M}(V', W)$ . It is known that  $Z(V, V', W)$  is lagrangian for type *ADE* (see [Na98, Th. 7.2]). Hence fundamental classes of irreducible components of  $Z(V, V', W)$  are examples of half-dimensional cycles.

**Example 16.** Consider the diagonal  $\Delta\mathfrak{M}(V, W)$  in  $\mathfrak{M}(V, W) \times \mathfrak{M}(V, W)$ . Its fundamental class gives an operator  $\Delta\mathfrak{M}(V, W) * \bullet: H_{d(V,W)}(\mathfrak{L}(V, W)) \rightarrow H_{d(V,W)}(\mathfrak{L}(V, W))$  by the above construction. It is the identity operator.

**4.4. Hecke correspondence.** Fix  $i \in Q_0$  and consider a pair  $V', V = V' \oplus S_i$  of  $Q_0$ -graded spaces, where  $S_i$  is 1-dimensional at  $i$  and 0 at other vertices. We define  $\mathfrak{P}_i(V, W) \subset \mathfrak{M}(V', W) \times \mathfrak{M}(V, W)$  consisting of points  $([B', I', J'], [B, I, J])$  such that  $[B', I', J']$  is a framed submodule of  $[B, I, J]$  ([Na98, §5]). More precisely, it means that there is an injective linear map  $\xi: V' \rightarrow V$  such that  $B\xi = \xi B', I\xi = I', J = J'\xi$ . Thus we have a short exact sequence of framed representations

$$(4.2) \quad 0 \rightarrow (B', I', J') \xrightarrow{\xi} (B, I, J) \rightarrow S_i \rightarrow 0,$$

where  $S_i$  is now regarded as a (simple) module with all linear maps are 0.

Let us explain the definition of operators for spaces of constructible functions. We have two projections  $p_1, p_2: \mathfrak{P}_i(V, W) \rightarrow \mathfrak{M}(V', W), \mathfrak{M}(V, W)$ . If  $f$  is a constructible function on  $\mathfrak{L}(V', W)$ , we pull back it to  $\mathfrak{P}_i(V, W) \cap p_1^{-1}(\mathfrak{L}(V', W))$  as  $p_1^*f = f \circ p_1$ . Then we define its pushforward  $p_{2!}(p_1^*f)$  defined by

$$(p_{2!}(p_1^*f))(x) = \sum_{a \in \mathbb{C}} a \chi(p_2^{-1}(x) \cap (p_1^*f)^{-1}(a)),$$

where  $\chi(\ )$  is the topological Euler number. This definition corresponds to (4.3) and we exchange roles of  $p_1, p_2$  for (4.4).

Let us explain the definition for homology groups. It was shown that  $\mathfrak{P}_i(V, W)$  is a smooth half-dimensional closed subvariety in  $\mathfrak{M}(V', W) \times \mathfrak{M}(V, W)$ . By its definition, it is contained in  $Z(V', V, W)$ . Thus the fundamental class  $[\mathfrak{P}_i(V, W)]$  defines an operator

$$(4.3) \quad [\mathfrak{P}_i(V, W)] * \bullet: H_{d(V',W)}(\mathfrak{L}(V', W)) \rightarrow H_{d(V,W)}(\mathfrak{L}(V, W)).$$

Changing the role of  $\mathfrak{M}(V, W), \mathfrak{M}(V', W)$ , we also have

$$(4.4) \quad [\mathfrak{P}_i(V, W)] * \bullet: H_{d(V,W)}(\mathfrak{L}(V, W)) \rightarrow H_{d(V',W)}(\mathfrak{L}(V', W)).$$

**4.5. Definition of Kac-Moody action.** Like (4.1) for  $\mathfrak{sl}(2)$ , a complex simple Lie algebra has a presentation given by generators  $E_i, F_i, H_i$  with certain relations. For an example, generators for  $\mathfrak{sl}(n+1)$  are  $E_i = E_{i,i+1}, F_i = E_{i+1,i}, H_i = E_{ii} - E_{i+1,i+1}$ , where  $E_{ij}$  is the matrix unit as before. For a Kac-Moody Lie algebra  $\mathfrak{g}$ , one needs to consider Cartan subalgebra  $\mathfrak{h}$ , which is larger than  $\text{Span}\{H_i\}$ . This is because we want to  $H_i$  to be linearly independent, even when the Cartan matrix has kernel. But this is basically just convention and is not so important. Let us ignore this difference, and defines action of  $E_i, F_i, H_i$  on the direct sum  $\bigoplus_V H_{d(V,W)}(\mathfrak{L}(V, W))$ .

Let

$$(4.5) \quad \begin{aligned} F_i &= (4.3), & E_i &= (-1)^{(d(V',W)-d(V,W))/2} \times (4.4), \\ H_i &= (\dim W_i - \sum_j a_{ij} \dim V_j) \operatorname{id}_{H_{d(V,W)}(\mathfrak{L}(V,W))}, \end{aligned}$$

where  $a_{ij}$  is the Cartan matrix, i.e.,  $2\delta_{ij} - \#\{h \in Q_1^{\text{dbl}} \mid \mathfrak{o}(h) = i, \mathfrak{i}(h) = j\}$ . Note that  $(\dim W_i - \sum_j a_{ij} \dim V_j)$  is the Euler characteristic of the complex (2.5) for  $(B^1, I^1, J^1) = (B, I, J)$ ,  $(B^2, I^2, J^2) = S_i$ , i.e.,  $(V^2, W^2) = (S_i, 0)$  with linear maps  $(B^2, I^2, J^2) = 0$ . This is a simple observation, and its brief explanation will be given below. It is even more important to consider (2.5) when one consider larger algebras action on homology/K-theory of quiver varieties.

**Theorem 17** ([Na94, Na98]). *Operators (4.5) satisfy the defining relations of the Kac-Moody Lie algebra  $\mathfrak{g}$ . Hence  $\bigoplus_V H_{d(V,W)}(\mathfrak{L}(V,W))$  is a representation of  $\mathfrak{g}$ . Moreover it is an (irreducible) integrable highest weight representation with the highest weight vector  $[\mathfrak{M}(0,W)] \in H_0(\mathfrak{M}(0,W))$ .*

When  $V = 0$ , the quiver variety  $\mathfrak{M}(0,W)$  is a single point as all linear maps  $B, I, J$  are automatically 0. As written above, this is the highest weight vector with highest weight  $\dim W$ , i.e., it satisfies

$$\begin{aligned} E_i[\mathfrak{M}(0,W)] &= 0, & H_i[\mathfrak{M}(0,W)] &= \dim W_i[\mathfrak{M}(0,W)] && \text{for all } i \in Q_0, \\ \mathbf{U}(\mathfrak{g})[\mathfrak{M}(0,W)] &= \bigoplus_V H_{d(V,W)}(\mathfrak{M}(V,W)), \end{aligned}$$

where  $\mathbf{U}(\mathfrak{g})$  is the universal enveloping algebra of  $\mathfrak{g}$ . The second condition, more concretely, means that the direct sum  $\bigoplus_V H_{d(V,W)}(\mathfrak{M}(V,W))$  is spanned by vectors obtained from  $[\mathfrak{M}(0,W)]$  by successively applying various  $F_i$ .

An integrability means that  $E_i, F_i$  are locally nilpotent, that is  $E_i^N m = 0 = F_i^N m$  for sufficiently large  $N = N(m)$  for a vector  $m$ . (For a complex simple Lie algebra, it is known to be equivalent to that the representation is finite dimensional.) It is known that an integrable highest weight representation is automatically irreducible.

Let us briefly explain the proof of the first part of Theorem 17. The most delicate relation to check is  $[E_i, F_j] = \delta_{ij} H_i$ . Once this is proved, the so-called Serre relation follows from it together with the integrability.

It is relatively easy to check the relation for  $i \neq j$ . For the proof of  $[E_i, F_i] = H_i$ , a key is to understand fibers of projections  $p_1, p_2: \mathfrak{P}(V,W) \rightarrow \mathfrak{M}(V',W), \mathfrak{M}(V,W)$ . By (4.2), the fiber of  $p_2$  at  $[B, I, J]$  is isomorphic to the projective space associated with the vector space  $\operatorname{Hom}((B, I, J), S_i)$ , where  $\operatorname{Hom}$  is the space of homomorphism as framed representations. This is the first cohomology of the complex (2.5) for  $(B^1, I^1, J^1) = (B, I, J)$ ,  $(B^2, I^2, J^2) = S_i$ . As we have remarked before, it is dual to the third cohomology of the complex (2.5) with  $(B^1, I^1, J^1)$  and  $(B^2, I^2, J^2)$  are swapped. On the other hand, the fiber of  $p_1$  at  $[B', I', J']$  is isomorphic to the projective space associated with  $\operatorname{Ext}^1(S_i, (B', I', J'))$ . This is the middle cohomology of the complex (2.5) for  $(V^1, W^1) = (S_i, 0)$ ,  $(V^2, W^2) = (V', W)$ . Then one observes that the complex (2.5) with  $(V^1, W^1) = (S_i, 0)$  has the vanishing first cohomology group if  $(B^2, I^2, J^2)$  satisfy the stability condition for  $\zeta_i^{\mathbb{R}} > 0$ . This is obvious as  $0 \neq \xi \in \operatorname{Ker} \alpha$  realizes  $S_i$  as a submodule of  $(B, I, J)$ . Then  $\zeta_i^{\mathbb{R}}(\dim S_i) = \zeta_i^{\mathbb{R}} > 0$



violates the stability condition. Thus the difference of dimensions of the second and third cohomology groups of (2.5) is the Euler characteristic of (2.5), hence can be computed. Now one uses that the Euler number of the complex projective space  $\mathbb{C}P^n$  is  $n + 1$  to complete the calculation.

**4.6. Inductive construction of irreducible components.** Let us sketch the proof of the second statement of Theorem 17. It is clear that  $E_i^N m = 0$  for sufficiently large  $N$ , as the dimension of  $V_i$  cannot be negative. For  $F_i^N m = 0$ , we use the vanishing of the first cohomology group of (2.5) for  $V^1 = S_i$ . If  $N$  is sufficiently large, the dimension of the first term exceeds that of the middle, hence  $\alpha$  cannot be injective.

Let us next explain why the representation is highest weight. It means that  $\bigoplus H_{d(V,W)}(\mathfrak{L}(V,W))$  is spanned by vectors obtained from  $[\mathfrak{M}(0,W)]$  by successively applying various  $F_i$ . This will be shown by an inductive construction of irreducible components of  $\mathfrak{L}(V,W)$ . In fact, it also gives Kashiwara crystal structure on the union of the set of irreducible components of  $\mathfrak{L}(V,W)$  with various  $V$ . Since it is not our purpose to review crystal bases, we do not explain this statement, and we concentrate only on the inductive construction.

Let us take  $[B, I, J] \in \mathfrak{M}(V,W)$  and consider (2.5) with  $(B^1, I^1, J^1) = S_i$ ,  $(B^2, I^2, J^2) = (B, I, J)$ , i.e.,

$$V_i \xrightarrow{\alpha} \bigoplus_{j \neq i} V_j^{\oplus -a_{ij}} \oplus W_i \xrightarrow{\beta} V_i.$$

As we noted, the first cohomology group vanishes. Consider the third cohomology group, which is the dual of the space  $\text{Hom}((B, I, J), S_i)$  of homomorphisms from  $(B, I, J)$  to  $S_i$ . We have a natural homomorphism  $(B, I, J) \rightarrow \text{Hom}((B, I, J), S_i)^\vee \otimes_{\mathbb{C}} S_i$ , which is given by the natural projection  $V_i \rightarrow \text{Cok } \beta$ . In particular, it is surjective. We consider the kernel of the natural homomorphism, and denote it by  $(B', I', J')$ . Thus we have

$$(4.6) \quad 0 \rightarrow (B', I', J') \rightarrow (B, I, J) \rightarrow \text{Hom}((B, I, J), S_i)^\vee \otimes S_i \rightarrow 0.$$

One can check that  $(B', I', J')$  is  $\zeta^{\mathbb{R}}$ -stable, hence defines a point in  $\mathfrak{M}(V', W)$  with  $\dim V' = \dim V - r \dim S_i$ , where  $r = \dim \text{Hom}((B, I, J), S_i)$ . Moreover we have the induced exact sequence

$$\text{Hom}((B, I, J), S_i) \otimes \text{Hom}(S_i, S_i) \rightarrow \text{Hom}((B, I, J), S_i) \rightarrow \text{Hom}((B', I', J'), S_i)$$

from the short exact sequence (4.6). The first homomorphism is an isomorphism by the construction. The second homomorphism is surjective as  $\text{Ext}^1(S_i, S_i) = 0$ . Therefore we conclude  $\text{Hom}((B', I', J'), S_i) = 0$ . It means that the complex

$$(4.7) \quad V_i' \xrightarrow{\alpha'} \bigoplus_{j \neq i} V_j'^{\oplus -a_{ij}} \oplus W_i \xrightarrow{\beta'} V_i'$$

has the vanishing third cohomology group.

Conversely we take  $(B', I', J')$  with  $\text{Hom}((B', I', J'), S_i) = 0$ . Then we recover  $(B, I, J)$  from an  $r$ -dimensional subspace in  $\text{Ext}^1(S_i, (B', I', J')) = \text{Ker } \beta' / \text{Im } \alpha'$ .

We use this construction to understand  $H_{d(V,W)}(\mathfrak{L}(V,W))$  as follows. (I learned this argument in [Lu90b].) Let  $Y$  be an irreducible component of  $\mathfrak{L}(V,W)$  with  $V \neq 0$ . We define  $\varepsilon_i(Y)$  be  $\dim \text{Hom}((B, I, J), S_i)$  for a generic  $[B, I, J] \in Y$ . From the nilpotency of  $(B, I, J)$ , there exists  $i \in Q_0$  such that  $\varepsilon_i(Y) > 0$ . Set  $r = \varepsilon_i(Y)$ . Then we

$Y^\circ = \{[B, I, J] \in Y \mid \dim \text{Hom}((B, I, J), S_i) = r\}$  is open in  $Y$ . We apply the above construction to  $[B, I, J] \in Y^\circ$  to obtain an irreducible variety  $Y'^\circ$  in  $\mathfrak{M}(V', W)$  with  $\dim V' = \dim V - r \dim S_i$ . It can be shown that its closure  $Y' = \overline{Y'^\circ}$  is an irreducible component of  $\mathfrak{L}(V', W)$ . In fact,  $Y' \subset \mathfrak{L}(V', W)$  is clear from the definition, as  $(B, I, J)$  and  $(B', I', J')$  have the same image under  $\pi$ . Next note that

$$d(V, W) - d(V', W) = 2r(\text{Euler characteristic of (4.7)} - r).$$

On the other hand,  $Y^\circ$  is the total space of Grassmann bundle of  $r$ -planes in the vector bundle over  $Y^\circ$  with fiber  $\text{Ext}^1(S_i, (B', I', J'))$ . Hence its dimension is equal to  $\dim Y'^\circ + r(\dim \text{Ext}^1(S_i, (B', I', J')) - r)$ . Since the Euler characteristic of (4.7) is  $\dim \text{Ext}^1(S_i, (B', I', J'))$ , we conclude that  $\dim Y'$  is half-dimensional in  $\mathfrak{M}(V', W)$ .

We deduce

$$\frac{F_i^r}{r!}[Y'] = \pm[Y] + \sum_{\varepsilon_i(Y'') > r} c_{Y''}[Y''] \quad c_{Y''} \in \mathbb{Q}.$$

By induction with respect to  $\dim V$  and  $\varepsilon_i$ , we get the assertion.

**Example 18.** Let us give an example of the induction of irreducible components. Let us consider the  $A_2$ -quiver with  $\dim V = (1, 2)$ ,  $\dim W = (1, 2)$ . We have an irreducible component  $Y$  with  $\varepsilon_2(Y) = 1$ , which is obtained from  $\mathfrak{L}(V', W)$  with  $\dim V' = (1, 0)$ , which is a single point. Nonzero maps in  $Y$  are

$$\begin{array}{ccc} V_1 = \mathbb{C} & \xleftarrow{B_{1,2}} & V_2 = \mathbb{C}^2 \\ J_1 \downarrow & & \downarrow J_2 \\ W_1 = \mathbb{C} & & W_2 = \mathbb{C}^2. \end{array}$$

We can normalize  $J_1 = 1$  by  $\text{GL}(V_1)$ , then we see that  $Y$  is  $\mathbb{C}P^2$  as  $B_{1,2} \oplus J_2$  defines 2-dimensional subspace in  $V_1 \oplus W_2 = \mathbb{C}^3$ .

Let us consider  $\varepsilon_1(Y)$ . For generic  $[B, I, J] \in Y$ , we have  $B_{1,2} \neq 0$ , hence  $\varepsilon_1(Y) = 0$ . We add 1-dimensional space at the vertex 1, and consider the irreducible component  $Y''$  of  $\mathfrak{L}(V'', W)$  with  $\dim V'' = (2, 2)$ . Over  $[B, I, J] \in Y$ , it is given by a 1-dimensional subspace in the middle cohomology of the complex

$$\mathbb{C} = V_1 \xrightarrow{0 \oplus J_1} V_2 \oplus W_1 = \mathbb{C}^2 \oplus \mathbb{C} \xrightarrow{(B_{1,2}, 0)} V_1 = \mathbb{C}.$$

If  $B_{1,2} \neq 0$ , the middle cohomology is 1-dimensional, hence the choice of a 1-dimensional subspace is unique. But note that there is a point  $B_{1,2} = 0$  in  $Y$ . Then the middle cohomology group is 2-dimensional, hence we have choices parametrized by  $\mathbb{C}P^1$ . This shows that  $Y''$  is the blowup of  $Y = \mathbb{C}P^2$  at the point  $B_{1,2} = 0$ . It also gives an example where  $\dim \text{Hom}((B, I, J), S_i)$  jumps at a special point in an irreducible component.

## REFERENCES

- [Bo16] T. Bozec, *Quivers with loops and generalized crystals*, *Compo. Math.*, **152** (2016), no. 10, 1999–2040.
- [CB00] W. Crawley-Boevey, *On the exceptional fibers of Kleinian singularities*, *American J. of Math.*, **122** (2000), no. 5, 1027–1037.
- [CB01] ———, *Geometry of the moment map for representations of quivers*, *Compositio Math.*, **126** (2001), no. 3, 257–293.

- [CH97] W. Crawley-Boevey and M.P. Holland, *Noncommutative deformations of Kleinian singularities*, Duke Math., **92** (1997), no. 3, 605–635.
- [Fu96] W. Fulton, *Young Tableaux: With Applications to Representation Theory and Geometry*, London Mathematical Society Student Texts, **35**, Cambridge University Press, Cambridge, 1997. x+260 pp
- [Gi12] V. Ginzburg, *Lectures on Nakajima’s quiver varieties*, in Geometric methods in representation Theory. I., 145–219, Sémin. Congr. **24-I**, Soc. Math. France, Paris, 2012.
- [Ki94] A.D. King,, *Moduli of representations of finite-dimensional algebras*, Quart. J. Math. Oxford Ser. (2), **45** (1994), no. 180, 515–530.
- [Ki16] A. Kirillov, Jr., *Quiver representations and quiver varieties*, Graduate Studies in Mathematics, **174**. American Mathematical Society, Providence, RI, 2016. xii+295 pp.
- [Kr89] P. B. Kronheimer, *The construction of ALE spaces as hyper-Kähler quotients*, J. Differential Geom. **29** (1989), no. 3, 665–683. MR MR992334 (90d:53055)
- [KN90] P. B. Kronheimer and H. Nakajima, *Yang-Mills instantons on ALE gravitational instantons*, Math. Ann. **288** (1990), no. 2, 263–307.
- [LP90] L. Le Bruyn and C. Procesi, *Semisimple representations of quivers*, Trans. Amer. Math. Soc **317** (1990), no. 2, 585–598.
- [Lu90a] G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), no. 2, 447–498.
- [Lu90b] ———, *Canonical bases arising from quantized enveloping algebras. II*, Progr. Theoret. Phys. Suppl., **102** (1990), 175–201.
- [MO12] D. Maulik and A. Okounkov, *Quantum groups and quantum cohomology*, arXiv:1211.1287.
- [Na94] H. Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*, Duke Math. **76** (1994), no. 2, 365–416.
- [Na96] ———, *Varieties associated with quivers*, in Representation theory of algebras and related topics (Mexico City, 1994), CMS Conf. Proc. **19** (1996), 139–157, Amer. Math. Soc.
- [Na98] ———, *Quiver varieties and Kac-Moody algebras*, Duke Math. **91** (1998), no. 3, 515–560.
- [Na01] ———, *Quiver varieties and tensor products*, Invent. Math. **146** (2001), no. 2, 399–449.
- [Ri90] C.M. Ringel, *Hall algebras and quantum groups*, Invent. Math. **101** (1990), no. 3, 583–591.
- [Sc08] O. Schiffmann, *Variété carquois de Nakajima (d’après Nakajima, Lusztig, Varagnolo, Vasserot, Crawley-Boevey et al.)*, Séminaire Boubaki Vol. 2006/2007, exposé no 976, Astérisque **317** (2008), 295–344.

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES  
 KYOTO UNIVERSITY  
 KYOTO, KYOTO 606-8502 JAPAN  
*E-mail address:* nakajima@kurims.kyoto-u.ac.jp