MUTATIONS OF SPLITTING MAXIMAL MODIFYING MODULES ARISING FROM DIMER MODELS

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Abstract. It is known that every 3-dimensional Gorenstein toric singularity has a crepant resolution. Although it is not unique, any crepant resolutions are connected by repeating the operation "flop". On the other hand, this singularity has a non-commutative crepant resolution (= NCCR) arising from a consistent dimer model. Such an NCCR is given as the endomorphism ring of a certain module which we call splitting maximal modifying module. In this article, we show all splitting maximal modifying modules are connected by repeating the operation "mutation" for some special cases.

1. Introduction

The notion of non-commutative crepant resolution (= NCCR) was introduced by Van den Bergh [16]. It is an algebra derived equivalent to crepant resolutions for a nice singularity, and it gives another perspective on Bondal-Orlov conjecture [4] and Bridgeland's theorem [5]. NCCRs are also related with Cohen-Macaulay representation theory (e.g., cluster tilting modules and their variants), and the present article rather stands on this viewpoint. Here, we recall the definition of NCCR [16].

Definition 1. Let R be a Cohen-Macaulay ring, and M be a non-zero reflexive R-module. Let $E := \mathrm{End}_R(M)$. We say E is a *non-commutative crepant resolution* (= *NCCR*) of R or M *gives an NCCR* of R if $\mathrm{gl.dim}\, E_{\mathfrak{p}} = \dim R_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Spec}\, R$ and E is a maximal Cohen-Macaulay R-module.

For example, an NCCR of a quotient singularity is given by the skew group algebra (see e.g. [16, 13]). Another interesting family of NCCRs is given by a dimer model. It gives us an NCCR of a 3-dimensional Gorenstein toric singularity under the "consistency condition" (see e.g. [6, 11, 2]). In the following, we consider this singularity. Thus, let Δ be a lattice polygon in $\mathbb{R}^2$ whose vertices are $v_1, \cdots, v_n \in \mathbb{Z}^2$. We define the cone σ whose section on the hyperplane $z = 1$ is Δ. That is, we add the third coordinate $z = 1$ to each vector v_i and define the cone σ as

$$\sigma = \mathrm{Cone}\{(v_1, 1), \cdots, (v_n, 1)\} = \mathbb{R}_{\geq 0}(v_1, 1) + \cdots + \mathbb{R}_{\geq 0}(v_n, 1) \subset \mathbb{R}^3.$$

Then, we consider the dual cone $\sigma^\vee := \{x \in \mathbb{R}^3 \mid \langle x, y\rangle \geq 0 \text{ for all } y \in \sigma\}$, where $\langle x, y\rangle$ is an inner product. This $\sigma^\vee \cap \mathbb{Z}^3$ is a positive affine normal semigroup, and hence we can define the *toric singularity* (or *toric ring*) R as

$$R := k[\sigma^\vee \cap \mathbb{Z}^3] = k[t_1^{a_1} t_2^{a_2} t_3^{a_3} \mid (a_1, a_2, a_3) \in \sigma^\vee \cap \mathbb{Z}^3].$$

The detailed version of this paper has been submitted for publication elsewhere.

It is known that a toric singularity defined by this manner is Gorenstein in dimension three. Conversely, any 3-dimensional Gorenstein toric singularity takes this form. We call this Δ the *toric diagram* of R, and call R *toric singularity associated with* Δ.

It is known that the ordinary crepant resolution of R corresponds to a triangulation of the toric diagram Δ. Thus, a crepant resolution of R exists but it is not unique in general. If there is a quadrangle consisting of two elementary triangles in a given triangulation, we obtain another triangulation by switching the diagonal, and it induces another crepant resolution. This operation is called flop and any crepant resolutions are connected by repeating this operation. On the other hand, as we mentioned, we obtain an NCCR of a 3-dimensional Gorenstein toric singularity from a dimer model (see Section 2). A *dimer model* is a polygonal cell decomposition of the real two-torus whose nodes and edges form a finite bipartite graph. We obtain a quiver with potential (= QP) as the dual of a dimer model, and the Jacobian algebra arising from such a QP will be an NCCR of a certain 3-dimensional Gorenstein toric singularity under the consistency condition. This algebra is isomorphic to the endomorphism ring of a reflexive module that is called a splitting maximal modifying (= MM) module. Conversely, for every 3-dimensional Gorenstein toric singularity, there exists a consistent dimer model giving an NCCR. Thus, an NCCR of R always exists, but a consistent dimer model which gives an NCCR of R is not unique in general. Hence, a splitting MM module is also not unique. Therefore, it is natural to observe a relationship between splitting MM modules. We recall that any crepant resolutions of R are connected by repeating the flop. How about splitting MM modules giving NCCRs ? Namely, is there a good operation that connects all splitting MM modules? In this article, we will consider the operation called "mutation", and show that all splitting MM modules are connected by repeating this operation for several cases (see Theorem 8). We note that this problem is open in general.

Conventions and Notations. Throughout, we assume that k is an algebraically closed field of characteristic zero. In this paper, we assume that all modules are left modules. For a ring R, we denote by $\mathsf{mod}R$ the category of finitely generated R-modules, denote by $\mathsf{add}_R M$ the full subcategory consisting of direct summands of finite direct sums of some copies of $M \in \mathsf{mod}R$, denote by $\mathsf{ref}R$ the category of reflexive modules, denote by $\mathsf{CM}R$ the category of maximal Cohen-Macaulay modules. We say that $M \in \mathsf{mod}R$ is a *generator* if $R \in \mathsf{add}_R M$. When we consider a composition of morphism, fg means we firstly apply f then g. With this convention, $\mathrm{Hom}_R(M, X)$ is an $\mathrm{End}_R(M)$-module and $\mathrm{Hom}_R(X, M)$ is an $\mathrm{End}_R(M)^{\mathrm{op}}$-module. Similarly, in a quiver, a path ab means a then b.

2. Dimer models and quivers with potentials

2.1. **What is a dimer model ?** In this subsection, we introduce the notion of dimer models. This notion was introduced in the field of statistical mechanics in 1960s. From 2000s, string theorists have been used it for studying quiver gauge theories. Subsequently, a dimer model has been investigated actively, and recently relations with many branches of mathematics (for example, the McKay correspondence, crepant resolutions, non-commutative crepant resolutions, Calabi-Yau algebras, mirror symmetry, etc) have been discovered. For more details, see e.g., [3].

A *dimer model* (or *brane tiling*) is a polygonal cell decomposition of the real two-torus $\mathsf{T} := \mathbb{R}^2/\mathbb{Z}^2$ whose nodes and edges form a finite bipartite graph. Therefore, we color each node either black or white, and each edge connects a black node to a white node. For a dimer model Γ, we denote the set of nodes (resp. edges, faces) of Γ by Γ_0 (resp. Γ_1, Γ_2). For example, the left hand side of Figure 1 is a dimer model where the outer frame is the fundamental domain of T.

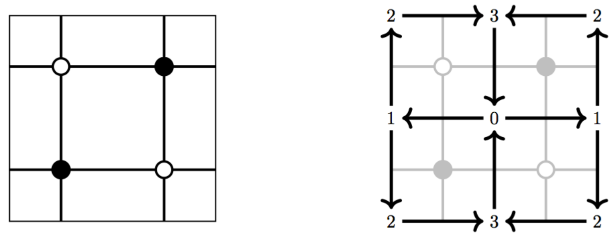

FIGURE 1

As the dual of a dimer model Γ, we define the quiver Q_Γ associated with Γ. Namely, we assign a vertex dual to each face in Γ_2, an arrow dual to each edge in Γ_1. The orientation of arrows is determined so that the white node is on the right of the arrow. For example, the right hand side of Figure 1 is the quiver obtained from the dimer model on the left. (Note that common numbers are identified in this figure.) Sometimes we simply denote the quiver Q_Γ by Q. We denote the set of vertices by Q_0 and the set of arrows by Q_1. We consider the set of oriented faces Q_2 as the dual of nodes on a dimer model Γ. The orientation of faces is determined by its boundary, that is, faces dual to white (resp. black) nodes are oriented clockwise (resp. anti-clockwise). Therefore, we decompose the set of faces as $Q_2 = Q_2^+ \sqcup Q_2^-$ where Q_2^+, Q_2^- denote the set of faces oriented clockwise and that of faces oriented anti-clockwise respectively. We define the maps $h, t : Q_1 \to Q_0$ that send an arrow $a \in Q_1$ to the head of a and the tail of a respectively. A nontrivial path is a finite sequence of arrows $a = a_1 \cdots a_r$ with $h(a_\ell) = t(a_{\ell+1})$ for $\ell = 1, \cdots r-1$. We consider each vertex $i \in Q_0$ as a trivial path e_i of length 0 where $h(e_i) = t(e_i) = i$. We extend the maps h, t to the maps on paths, that is, $t(a) = t(a_1), h(a) = h(a_r)$. We say that a path a is a *cycle* if $h(a) = t(a)$. For this quiver Q, let kQ be the *path algebra*. That is, kQ is a k-algebra whose k-basis consists of paths in Q and the product of basis elements is defined as $a \cdot b = ab$ (resp. $a \cdot b = 0$) if $h(a) = t(b)$ (resp. $h(a) \neq t(b)$) for paths a and b, and we extend this product linearly. In addition, we define a certain potential. We denote by $[kQ, kQ]$ the k-vector space generated by all commutators in kQ and set the vector space $kQ_{\mathrm{cyc}} := kQ/[kQ, kQ]$, thus kQ_{cyc} has a basis consists of cycles in Q. For each face $f \in Q_2$, we associate the *small cycle* $\omega_f \in KQ_{\mathrm{cyc}}$ obtained as the product of arrows around the boundary of f. For the quiver Q associated with a dimer model, we define the potential W_Q as

$$W_Q := \sum_{f \in Q_F^+} \omega_f - \sum_{f \in Q_F^-} \omega_f.$$

We call a pair (Q, W_Q) a *quiver with potential* ($= QP$). For each face $f \in Q_2$, we choose an arrow $a \in \omega_f$ and consider $h(a)$ as the starting point of the small cycle ω_f. Then we set $e_{h(a)}\omega_f e_{h(a)} := a_1 \cdots a_r a$ for some path $a_1 \cdots a_r$. We define the partial derivative of ω_f with respect to a by $\partial\omega_f/\partial a := a_1 \cdots a_r$. Extending this derivative linearly, we also define $\partial W_Q/\partial a$ for any $a \in Q_1$. We define the two-sided ideal $J(W_Q) := \langle \partial W_Q/\partial a \,|\, a \in Q_1 \rangle$. Then, we define the *Jacobian algebra* of Q as

$$\mathcal{P}(Q, W_Q) := kQ/J(W_Q).$$

From this construction, $\partial W_Q/\partial a$ gives a relation on paths in $\mathcal{P}(Q, W_Q)$ for each arrow $a \in Q_1$. Namely, for each arrow $a \in Q_1$, there are precisely two oppositely oriented faces containing the arrow a as a boundary. We denote them by $f_a^+, f_a^- \in Q_2$ respectively. Let $p_a^\pm$ be the path from $h(a)$ around the boundary of $f_a^\pm$ to $t(a)$. Then we can describe $\partial W_Q/\partial a$ as a difference of $p_a^\pm$, that is, $\partial W_Q/\partial a = p_a^+ - p_a^-$. Thus, we have $p_a^+ = p_a^-$ in $\mathcal{P}(Q, W_Q)$ for each arrow $a \in Q_1$.

In the rest, we assume a dimer model has no bivalent nodes which are nodes connecting only two distinct nodes. If there are bivalent nodes, we remove them as shown in [10, Figure 5.1] because this operation does not change the Jacobian algebra up to isomorphism.

2.2. **Consistency conditions and NCCRs.** In this subsection, we introduce the consistency condition. Under this assumption, a dimer model gives an NCCR (see Theorem 3). In the literature, there are several consistency conditions, see e.g., [1, 10], and almost all conditions are equivalent. Here, we note one of them.

Definition 2. Let Q be a quiver associated with a dimer model Γ. A positively grading $\mathsf{R} : Q_1 \to \mathbb{R}_{>0}$ satisfying the following conditions is called a *consistent* R*-charge*.

(1) $\displaystyle\sum_{a \in \partial f} \mathsf{R}(a) = 2$ for any $f \in Q_2$,

(2) $\displaystyle\sum_{h(a)=i} (1 - \mathsf{R}(a)) + \sum_{t(a)=i} (1 - \mathsf{R}(a)) = 2$ for any $i \in Q_0$.

We say that a dimer model Γ is *consistent* if it admits a consistent R-charge.

Under this assumption, we obtain an NCCR of a 3-dimensional Gorenstein toric singularity.

Theorem 3. *(see e.g.,* [6, 11, 2]*) Let (Q, W_Q) be the QP associated with a consistent dimer model and $\mathcal{P}(Q, W_Q)$ be the Jacobian algebra. Then, the center of $\mathcal{P}(Q, W_Q)$ is a 3-dimensional Gorenstein toric singularity $R := \mathrm{Z}(\mathcal{P}(Q, W_Q))$. Moreover, there exists a reflexive module M satisfying*

$$\mathcal{P}(Q, W_Q) \cong \mathrm{End}_R(M),$$

and this is an NCCR of R.

In this way, we obtain 3-dimensional Gorenstein toric singularities and their NCCRs from consistent dimer models. Conversely, for every 3-dimensional Gorenstein toric singularity R, there exists a consistent dimer model giving R as the center of the Jacobian algebra [8, 11]. Therefore, every 3-dimensional Gorenstein toric singularity admits an NCCR arising from a consistent dimer model. However, we remark that a consistent dimer model giving an NCCR of R is not unique in general. For more details, see the survey article [3].

Next, we discuss a reflexive module M giving an NCCR of R. In our situation, it is known that reflexive modules giving NCCRs of a 3-dimensional Gorenstein toric singularity R are precisely maximal modifying modules [13]. Here, we say that $M \in \mathsf{ref} R$ is a *maximal modifying module* (= *MM module*) if $\mathrm{End}_R(M) \in \mathsf{CM} R$ and if there exists $X \in \mathsf{ref} R$ satisfying $\mathrm{End}_R(M \oplus X) \in \mathsf{CM} R$ then $X \in \mathsf{add}_R M$. Furthermore, if the endomorphism ring of an MM module M is isomorphic to the Jacobian algebra arising from a consistent dimer model, then M is a finite direct sum of rank one reflexive modules. We call such an MM module M a *splitting MM module*. Conversely, we also see that for each splitting MM module M, there is a QP (Q, W_Q) associated with a consistent dimer model satisfying $\mathrm{End}_R(M) \cong \mathcal{P}(Q, W_Q)$. Thus, we have the following.

Corollary 4. *For a 3-dimensional Gorenstein toric singularity R, splitting MM modules are precisely modules giving NCCRs of R arising from consistent dimer models.*

Especially, we have a bijection between vertices of Q and direct summands in a basic splitting MM module M. Thus, we may write $M = \oplus_{i \in Q_0} M_i$, and an isomorphism $\mathcal{P}(Q, W_Q) \cong \mathrm{End}_R(M)$ is obtained by sending a path $(i \to j)$ to a morphism $(M_i \to M_j)$.

In this manner, we obtain NCCRs of a 3-dimensional Gorenstein toric singularity R arising from consistent dimer models via splitting MM modules. Since a consistent dimer model giving an NCCR of R is not unique, a splitting MM module is also not unique. Thus, it is natural to ask the following, and we discuss this question in the next section.

Question 5. *Let $M_1, \cdots, M_s$ be splitting MM modules of a 3-dimensional Gorenstein toric singularity R giving NCCRs arising from consistent dimer models. Is there a relationship between these splitting MM modules ?*

3. Mutations of splitting MM generators

In the rest, we suppose $R = K[[t_1^{\alpha_1} t_2^{\alpha_2} t_3^{\alpha_3} \mid (\alpha_1, \alpha_2, \alpha_3) \in \sigma^\vee \cap \mathbb{Z}^3]]$ is the $\mathfrak{m}$-adic completion of a 3-dimensional Gorenstein toric singularity where $\mathfrak{m}$ is the irrelevant maximal ideal. Note that we can obtain this singularity as the center of the complete Jacobian algebra of a certain dimer model. For simplicity, we will only consider splitting MM generators in the rest of this article. Let M be a splitting MM generator. As we mentioned, we can obtain a consistent dimer model Γ and the associated QP (Q, W_Q) satisfying $\mathcal{P}(Q, W_Q) \cong \mathrm{End}_R(M)$, and we may write $M = \bigoplus_{i \in Q_0} M_i$, especially we set $M_0 = R$. In order to consider Question 5, we consider the mutation of a splitting MM generator M at $0 \neq k \in Q_0$, which is denoted by $\mu_k(M)$. By the results in [13], $\mu_k(M)$ is also an MM module and gives an NCCR. However, we remark that even if M is a splitting MM generator, $\mu_k(M)$ is not a splitting MM generator in general. In order to make $\mu_k(M)$ a splitting MM generator, we need the following condition ($\clubsuit$) on $k \in Q_0$, and we will denote the subset of Q_0 satisfying ($\clubsuit$) by Q_0^μ (for more details, see [14, Section 4]).

($\clubsuit$) $k \in Q_0$ has exactly two incoming arrows (thus has exactly two outgoing arrows).

Now, let us define the mutation of M at $0 \neq k \in Q_0^\mu$. First, we suppose that $a_1, a_2 \in Q_1$ are two incoming arrows (i.e. $h(a_1) = h(a_2) = k$) and set $N_k := \bigoplus_{i \in Q_0 \setminus \{k\}} M_i$. Then, we can take a minimal right $(\mathsf{add}_R N_k)$-approximation $\varphi : M_{t(a_1)} \oplus M_{t(a_2)} \to M_k$. That is, it is a morphism satisfying:

- $\mathrm{Hom}_R(N_k, M_{t(a_1)} \oplus M_{t(a_2)}) \xrightarrow{\cdot\varphi} \mathrm{Hom}_R(N_k, M_k)$ is surjective,
- if $\phi \in \mathrm{End}_R(M_j \oplus M_k)$ satisfies $\phi\varphi = \varphi$, then ϕ is an automorphism.

We denote by $\mathcal{K}_k$ the kernel of this morphism. Then, the *mutation* of M at k is defined as $\mu_k(M) = \bigoplus_{i \in Q_0 \setminus \{k\}} M_i \oplus \mathcal{K}_k$. By counting the rank, we see that $\mathrm{rank}_R \mathcal{K}_k = 1$, hence $\mu_k(M)$ is a splitting MM generator.

In the following, we consider Question 5 using this mutation. Especially, we observe the exchange graph of splitting MM generators. That is, let $\mathrm{MMG}_1(R)$ be the set of isomorphism classes of basic splitting MM generators of R, and $\mathsf{EG}(\mathrm{MMG}_1(R))$ be the graph whose vertices are elements in $\mathrm{MMG}_1(R)$, where we draw an edge between M and $\mu_k(M)$ for each $M \in \mathrm{MMG}_1(R)$ and $0 \neq k \in Q_0^\mu$. In what follows, we will show that $\mathsf{EG}(\mathrm{MMG}_1(R))$ is connected for some special cases (other cases are still open).

As we will see in examples below, we easily describe the exchange graph for the case of simplicial cones and the A_1-singularity (or conifold). Especially, we see that they are connected.

Example 6. Let Δ be a triangle polygon. Then the associated cone is simplicial, and hence toric singularity R associated with Δ is a quotient singularity by a finite abelian group $G \subset \mathrm{SL}(3, k)$. In this case, R has a unique basic splitting MM generator (see [12, Theorem 3.1]). Especially, the exchange graph of splitting MM generators is a single point, and there is a unique consistent dimer model giving such a splitting MM generator. Note that the associated quiver is the McKay quiver of G. (For more details, see [15], [12, Corollary 1.7].)

Example 7. Let R be the 3-dimensional A_1-singularity (i.e. $R \cong K[[x, y, z, w]]/(xy - zw)$). Remark that R is of finite CM representation type, that is, it has only finitely many non-isomorphic indecomposable MCM modules, and finitely many MCM R-modules are R, $I = (x, z)$, $I^* = (x, w)$ (see e.g. [17]). Then modules giving an NCCR are only $R \oplus I$ and $R \oplus I^*$ [16], and they are splitting MM generators. By taking a minimal right $\mathsf{add}R$-approximation of I: $0 \to I^* \to R^2 \to I \to 0$, we can connect $R \oplus I$ to $R \oplus I^*$ in $\mathsf{EG}(\mathrm{MMG}_1(R))$. Also, for these splitting MM generators, there is a unique consistent dimer model Γ such that $\mathcal{P}(Q_\Gamma, W_{Q_\Gamma}) \cong \mathrm{End}_R(R \oplus I) \cong \mathrm{End}_R(R \oplus I^*)$.

4. Splitting MM generators associated with reflexive polygons

In this section, we consider Question 5 for the case of 3-dimensional Gorenstein toric singularities associated with reflexive polygons. We say that Δ is a *reflexive polygon* (or *Fano polygon*) if the origin is a unique interior point of Δ. Reflexive polygons are classified in 16 types up to integral unimodular transformations (see e.g. [7, Theorem 8.3.7], [2, Appendix]). Consistent dimer models giving a toric singularity associated with a reflexive polygon are well-studied in several papers (see e.g. [2, 9]), and such dimer models are classified up to right equivalence of the associated QPs. Thus, we can obtain all splitting MM generators from those consistent dimer models. For such singularities, we have the connectedness of the exchange graph of splitting MM generators as follows. Furthermore, we can generalize this to the case of splitting MM modules (see [14, Section 6]).

Theorem 8. *(see* [14, Theorem 5.1]*) Let R be a 3-dimensional complete local Gorenstein toric singularity associated with a reflexive polygon. Then any two splitting MM generators are transformed into each other by repeating the mutation of splitting MM generators. Especially, the exchange graph of splitting MM generators is connected.*

The proof is a case-by-case check for all classified dimer models, and the strategy is similar for each type. Thus, we show one of them as an example.

Example 9. We consider the reflexive polygon whose vertices are $(1,0),(0,1),(-1,0),(0,-1)$. Thus, let R be the 3-dimensional complete local Gorenstein toric singularity defined by the cone σ:

$$\sigma = \mathrm{Cone}\{v_1 = (1,0,1), v_2 = (0,1,1), v_3 = (-1,0,1), v_4 = (0,-1,1)\}.$$

There are two consistent dimer models written below which give R as the center of the Jacobian algebra.

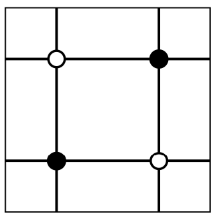

FIGURE 2

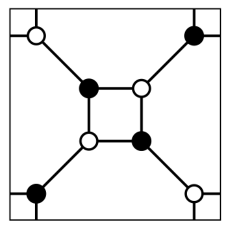

FIGURE 3

In general, rank one reflexive modules form the group called the class group $\mathrm{Cl}(R)$. In this case, we have that $\mathrm{Cl}(R) \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and hence each rank one reflexive module is represented by $(a,b) \in \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

First, we consider the consistent dimer model shown in Figure 2 and the associated Jacobian algebra A. In this case, there are two splitting MM generators whose endomorphism ring is isomorphic to A. Let $T(a,b)$ be a rank one reflexive module represented by (a,b) as the element in $\mathrm{Cl}(R)$, especially $T(0,0) = R$. Then, such two splitting MM generators are

$$R \oplus T(0,1) \oplus T(1,1) \oplus T(-1,0), \text{ and } R \oplus T(1,0) \oplus T(1,1) \oplus T(2,1)$$

Similarly, for the consistent dimer model shown in Figure 3, there are four splitting MM generators which are represented by

$$\begin{array}{ll} R \oplus T(1,0) \oplus T(2,1) \oplus T(3,1), & R \oplus T(0,1) \oplus T(1,0) \oplus T(1,1), \\ R \oplus T(0,1) \oplus T(-1,0) \oplus T(-1,1), & R \oplus T(-1,0) \oplus T(1,1) \oplus T(2,1). \end{array}$$

Using these modules, we describe the exchange graph $\mathsf{EG}(\mathrm{MMG}_1(R))$ as shown in Figure 4, and this is actually connected. In this figure, a double circle stands for the origin $(0,0) \in \mathbb{Z}^2$ and each point $(a,b) \in \mathbb{Z}^2$ corresponds to the R-module $T(a,b)$.

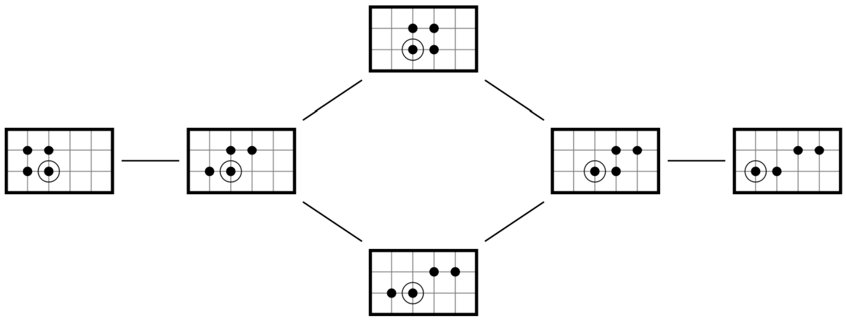

Figure 4. The exchange graph of splitting MM generators

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