LOCAL DUALITY PRINCIPLE AND GROTHENDIECK'S VANISHING THEOREM

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Abstract. Let R be a commutative Noetherian ring. In this article, we introduce the notion of local cohomology functor γ_W with support in a general subset W of $\operatorname{Spec} R$, which is a natural generalization of ordinary local cohomology functors supported in specialization-closed subsets. We propose a general principle behind the local duality theorem, and report that the vanishing theorem of Grothendieck type holds for γ_W.

1. Introduction

This article is based on a joint work with Yuji Yoshino [4].

Let R be a commutative Noetherian ring. We denote by $\mathcal{D} = D(\mathrm{Mod}\ R)$ the derived category of unbounded chain complexes of R-modules. Note that chain complexes X are cohomologically indexed;

$$X = (\cdots \to X^{i-1} \to X^i \to X^{i+1} \to \cdots).$$

We denote by $\mathcal{D}^+$ the full subcategory of $\mathcal{D}$ consisting of left bounded complexes. We also denote by $\mathcal{D}^-_{\mathrm{fg}}$ the full subcategory of $\mathcal{D}$ consisting of right bounded complexes with finitely generated cohomology modules.

For a chain complex X in $\mathcal{D}$, the small support of X is a subset of $\operatorname{Spec} R$ defined as

$$\operatorname{supp} X = \{\, \mathfrak{p} \in \operatorname{Spec} R \mid X \otimes^{\mathrm{L}}_R \kappa(\mathfrak{p}) \neq 0 \,\}.$$

Here we use general convention to denote $\kappa(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. In order to compare with the ordinary support, recall that the (big) support $\operatorname{Supp} X$ is a set of primes $\mathfrak{p}$ of R satisfying $X_\mathfrak{p} \neq 0$ in $\mathcal{D}$. In general, we have $\operatorname{supp} X \subseteq \operatorname{Supp} X$ and equality holds if $X \in \mathcal{D}^-_{\mathrm{fg}}$.

Now we recall that a full subcategory $\mathcal{L}$ of $\mathcal{D}$ is said to be a localizing subcategory if $\mathcal{L}$ is a triangulated subcategory of $\mathcal{D}$ and $\mathcal{L}$ is closed under making arbitrary direct sums. If we are given a subset W of $\operatorname{Spec} R$, then, since the tensor product commutes with taking direct sums, it is easy to see that the full subcategory $\mathcal{L}_W = \{\, X \in \mathcal{D} \mid \operatorname{supp} X \subset W \,\}$ is localizing. Conversely, a theorem of Neeman [5] ensures that any localizing subcategory of $\mathcal{D}$ is obtained in this way from a subset W of $\operatorname{Spec} R$.

If A is a set of objects in $\mathcal{D}$, then we denote by $\operatorname{Loc} A$ the smallest localizing subcategory of $\mathcal{D}$ containing all objects of A. We also denote by $E_R(R/\mathfrak{p})$ the injective envelope of the R-module $R/\mathfrak{p}$ for $\mathfrak{p} \in \operatorname{Spec} R$. Then it is easy to see $\operatorname{supp} \kappa(\mathfrak{p}) = \operatorname{supp} E_R(R/\mathfrak{p}) = \{\mathfrak{p}\}$. Moreover, in the above-mentioned paper [5], Neeman shows the equalities

$$\mathcal{L}_W = \operatorname{Loc}\{\, \kappa(\mathfrak{p}) \mid \mathfrak{p} \in W \} = \operatorname{Loc}\{\, E_R(R/\mathfrak{p}) \mid \mathfrak{p} \in W \,\}.$$

The detailed version of this paper will be submitted for publication elsewhere.

In general, for a localizing subcategory $\mathcal{L}$ of $\mathcal{D}$, its right orthogonal subcategory is defined as

$$\mathcal{L}^{\perp} = \{\, Y \in \mathcal{D} \,|\, \mathrm{Hom}_{\mathcal{D}}(X, Y) = 0 \text{ for all } X \in \mathcal{L} \,\}.$$

Note that $\mathcal{L}^{\perp}$ is a triangulated subcategory of $\mathcal{D}$ that is closed under arbitrary direct products. By this property $\mathcal{L}^{\perp}$ is called a colocalizing subcategory of $\mathcal{D}$.

By a classical argument of the localization theory of triangulated categories, we can show the following fact.

Lemma 1. *Let W be any subset of* $\mathrm{Spec}\, R$*. Then the inclusion functor $\mathcal{L}_W \hookrightarrow \mathcal{D}$ has a right adjoint γ_W, and the inclusion functor $\mathcal{L}_W^{\perp} \hookrightarrow \mathcal{D}$ has a left adjoint λ_W. For any object X of $\mathcal{D}$, there is a triangle*

$$\gamma_W X \to X \to \lambda_W X \to \gamma_W X[1].$$

Furthermore the triangle is unique up to isomorphisms in the following sense: If

$$X' \to X \to X'' \to X'[1]$$

is a triangle with $X' \in \mathcal{L}_W$ and $X'' \in \mathcal{L}_W^{\perp}$, then $X' \cong \gamma_W X$ and $X'' \cong \lambda_W X$.

Definition 2. Let W be a subset of $\mathrm{Spec}\, R$. Then we define the local cohomology functor with support in W as the functor $\gamma_W : \mathcal{D} \to \mathcal{L}_W$ that is right adjoint to the inclusion functor $\mathcal{L}_W \hookrightarrow \mathcal{D}$.

Recall that a subset W of $\mathrm{Spec}\, R$ is called specialization-closed if the following condition holds:

(*) Let $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}\, R$. If $\mathfrak{p} \in W$ and $\mathfrak{p} \subset \mathfrak{q}$, then $\mathfrak{q}$ belongs to W.

If W is a specialization-closed subset, then the local cohomology functor γ_W coincides with the ordinary local cohomology functor $\mathrm{R}\Gamma_W$ that is a right derived functor of the section functor with support in W, see [3, Appendix 3.5].

To describe the main theorem of this article, we introduce the following notion.

Definition 3. For a subset W of $\mathrm{Spec}\, R$, we denote by $\dim W$ the maximal length of chains of prime ideals belonging to W, i.e.

$$\dim W = \sup\{\, n \,|\, \text{there are } \mathfrak{p}_0, \dots, \mathfrak{p}_n \text{ in } W \text{ with } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \,\}.$$

The following result is the main theorem, which we call the *Local Duality Principle.*

Theorem 4 (Local Duality Principle)**.** *Let W be a subset of* $\mathrm{Spec}\ R$*. Assume that* $\dim W$ *is finite. Then there exists a canonical isomorphism*

$$\gamma_W \mathrm{RHom}_R(X, Y) \cong \mathrm{RHom}_R(X, \gamma_W Y)$$

for $X \in \mathcal{D}^{-}_{\mathrm{fg}}$ and $Y \in \mathcal{D}^{+}$.

This theorem generalizes Foxby's result [1, Proposition 6.1], which states the validity of the same isomorphism as above when W is a specialization-closed subset of $\mathrm{Spec}\ R$.

The Local Duality Principle naturally implies the following corollary.

Corollary 5. *Assume that R admits a dualizing complex D_R. Let W be an arbitrary subset of* $\operatorname{Spec} R$ *and X be a left bounded complex with finitely generated cohomology modules. We write $X^\dagger = \mathrm{RHom}_R(X, D_R)$. Then we have a natural isomorphism*

$$\gamma_W X \cong \mathrm{RHom}_R(X^\dagger, \gamma_W D_R).$$

If $(R, \mathfrak{m})$ is a Noetherian local ring endowed with a dualizing complex, and $W = \{\mathfrak{m}\}$, then the same isomorphism as above is the ordinary local duality theorem, see [2].

By Corollary 5, we can show the following theorem, which is Grothendieck type vanishing theorem for γ_W.

Theorem 6. *We assume that R admits a dualizing complex. Let W be a subset of* $\operatorname{Spec} R$ *and M be a finitely generated R-module. We write $H^i_W(M) = H^i(\gamma_W M)$. Then $H^i_W(M) = 0$ for $i > \dim M$.*

Now we shall explain an outline of the proof of the Local Duality Principle. We need the following notion.

Definition 7. Let $W_1 \subset W$ be subsets of $\operatorname{Spec} R$. We say that W_1 is *specialization-closed in W* if $V(\mathfrak{p}) \cap W \subseteq W_1$ for any $\mathfrak{p} \in W_1$.

Lemma 8. *Assume* $\dim W = 0$. *Let I be an injective R-module. Then we have the following isomorphism;*

$$\gamma_W I \cong \bigoplus_{\mathfrak{p} \in W} E(R/\mathfrak{p})^{\oplus B},$$

where $B = \dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_R(\kappa(\mathfrak{p}), I)$.

By the lemma above, we see that γ_W sends injective modules to injective modules if $\dim W = 0$.

The following theorem enables us to compute γ_W by using induction on $\dim W$.

Theorem 9. *Let $W_1 \subset W \subset \operatorname{Spec} R$ be sets. Assume that W_1 is specialization-closed in W, and set $W_0 = W \setminus W_1$. Then there is a triangle of the following form for any $X \in \mathcal{D}$;*

$$\gamma_{W_1} X \to \gamma_W X \to \gamma_{W_0} \lambda_{W_1} X \to \gamma_{W_1} X[1].$$

We denote by $\mathcal{K} = K(\operatorname{Inj} R)$ the homotopy category of chain complexes of injective R-modules. Let a and b be taken from $\mathbb{Z} \cup \{+\infty\}$, and we assume that $a \leq b$. We denote by $\mathcal{K}^{[a,b]}$ the full subcategory of $\mathcal{K}$ consisting of all chain complexes whose nontrivial ith components appear only in the interval $[a, b]$.

Corollary 10. *Assume that $d = \dim W$ is finite. Let $a, b \in \mathbb{Z} \cup \{+\infty\}$ with $a \leq b$. Then γ_W maps objects of $\mathcal{K}^{[a,b]}$ to objects of $\mathcal{K}^{[a-d,b]}$, i.e.*

$$\gamma_W(\mathcal{K}^{[a,b]}) \subseteq \mathcal{K}^{[a-d,b]}.$$

In particular, we have $\gamma_W(\mathcal{D}^+) \subset \mathcal{D}^+$ if $\dim W$ *is finite.*

Sketch of the proof of Theorem 4: Let $X \in \mathcal{D}^-_{\mathrm{fg}}$ and $Y \in \mathcal{D}^+$. Let W be a subset of $\operatorname{Spec} R$. Assume that $\dim W$ is finite. Applying the functor $\mathrm{RHom}_R(X, -)$ to the triangle

$$\gamma_W Y \to Y \to \lambda_W Y \to \gamma_W Y[1],$$

we obtain a triangle of the form;

$$\mathrm{RHom}_R(X, \gamma_W Y) \to \mathrm{RHom}_R(X, Y) \to \mathrm{RHom}_R(X, \lambda_W Y) \to \mathrm{RHom}_R(X, \gamma_W Y)[1].$$

Hence it follows from Lemma 1 that, to prove the desired isomorphism, we only have to show that

$$\mathrm{RHom}_R(X, \gamma_W Y) \in \mathcal{L}_W \quad \text{and} \quad \mathrm{RHom}_R(X, \lambda_W Y) \in \mathcal{L}_W^{\perp}.$$

By using tensor-hom adjunction, we can see that $\mathrm{RHom}_R(X, \lambda_W Y) \in \mathcal{L}_W^{\perp}$. On the other hand, thanks to Corollary 10, we can show that $\mathrm{RHom}_R(X, \gamma_W Y) \in \mathcal{L}_W$. For the details, see [4]. □

At the end of this article, we shall give a nontrivial examples of local cohomology functors γ_W, for which $\gamma_W I$ has a non-zero negative cohomology module even for an injective R-module I.

Example 11. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of dimension d. It is well-known that the $\mathfrak{m}$-adic completion $\widehat{R}$ possesses a dualizing complex, which can be described as follows;

$$D_{\widehat{R}} = (\ 0 \to I^0 \to I^1 \to \cdots \to I^{d-1} \to I^d \to 0\ )$$

where each term module I^n is an injective $\widehat{R}$-module of the form

$$I^n = \bigoplus_{\dim \widehat{R}/\mathfrak{P} = d-n} E_{\widehat{R}}(\widehat{R}/\mathfrak{P})$$

for $n = 0, 1, \cdots d$. In particular, we see that $I^d = E_{\widehat{R}}(k) = E_R(k)$. Considering a complex

$$J = \big(0 \to I^0 \to I^1 \to \cdots \to I^{d-1} \to 0\big)$$

we have $\gamma_{\{\mathfrak{m}\}^c} E_R(k) = J[d-1]$, where $\{\mathfrak{m}\}^c = \operatorname{Spec} R \backslash \{\mathfrak{m}\}$.

This example shows that γ_W is not necessarily a right derived functor of an additive functor defined on $\mathrm{Mod}R$.

References

[1] H.-B. Foxby, *Bounded complexes of flat modules*, J. Pure Appl. Algebra **15** (1979), 149–172.

[2] R. Hartshorne, *Residues and Duality: Lecture Notes of a Seminar on the Work of A. Grothendieck*, Lecture Notes in Math. **20**, Springer- Verlag, Berlin-New York (1966).

[3] J. Lipman, *Lectures on local cohomology and duality, Local cohomology and its applications*, Lect. Notes Pure Appl. Math. **226** (2002), Dekker, New York, 39–89.

[4] T. Nakamura and Y. Yoshino, *Local duality principle in derived categories*, arXiv:1610.07715v.

[5] A. Neeman, *The chromatic tower of $D(R)$*, Topology **31** (1992), 519–532.

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