WALL-CROSSING BETWEEN STABLE AND CO-STABLE ADHM DATA

RYO OHKAWA

Abstract. We prove formula between Nekrasov partition functions defined from stable and co-stable ADHM data for the plane following method by Nakajima-Yoshioka [4] based on the theory of wall-crossing formula developed by Mochizuki [2]. This formula is similar to conjectures by Ito-Maruyoshi-Okuda [1, (4.1), (4.2)] for A_1 singularity.

1. Introduction

Let Q be the affine plane $\mathbb{C}^2$, and $\mathbb{P}^2 = \mathbb{P}(\mathbb{C} \oplus Q)$ the projective plane. For the homogeneous coordinate $[x_0, x_1, x_2]$ of $\mathbb{P}^2$, we put $\ell_\infty = \{x_0 = 0\} \subset \mathbb{P}^2$ the infinity line.

Definition 1. A framed sheaf is a pair (E, Φ) of

- E : (torsion free) sheaf on $\mathbb{P}^2$,
- $\Phi\colon E|_{\ell_\infty} \cong \mathcal{O}_{\ell_\infty}^{\oplus r}$ an isomorphism, called framing.

We consider the moduli $M(r, n)$ of framed sheaves (E, Φ), $\mathrm{rk} E = r, c_2(E) = n$. In this report, we describe $M(r, n)$ in terms of *ADHM data*, and show functional equations for generating functions of integrations over $M(r, n)$, called *Nekrasov partition functions*.

2. ADHM data

Let $W = \mathbb{C}^r, V = \mathbb{C}^n$ be $\mathbb{C}$-vector spaces.

Definition 2. ADHM data are collections of linear maps (B_1, B_2, z, w)

$$B_1, B_2\colon V \to V, z\colon W \to V, w\colon V \to W$$

satisfying the condition $B_1B_2 - B_2B_1 + zw = 0$.

We put $[B_1, B_2] := B_1B_2 - B_2B_1$, and call $[B_1, B_2] + zw = 0$ *ADHM relation*.

Definition 3. ADHM data (B_1, B_2, z, w) are said to be stable if for any $S \subset V$ with $B_1(S), B_2(S) \subset S$ and $\mathrm{im}\, z \subset S$, we have $S = V$.

From stable ADHM data (B_1, B_2, z, w), we construct framed sheaves(E, Φ) as follows. Consider the following complex $C^\bullet$:

$$C^\bullet := 0 \to V \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\sigma} \begin{matrix} V \otimes \mathcal{O}_{\mathbb{P}^2} \\ \oplus \\ W \otimes \mathcal{O}_{\mathbb{P}^2} \end{matrix} \xrightarrow{\tau} V \otimes \mathcal{O}_{\mathbb{P}^2}(1) \to 0$$

The detailed version of this paper has been submitted for publication elsewhere.

Here

$$\sigma = \begin{bmatrix} -B_2 x_0 + \mathrm{id}_V\, x_2 \\ B_1 x_0 - \mathrm{id}_V\, x_1 \\ w x_0 \end{bmatrix}, \tau = \begin{bmatrix} B_1 x_0 - \mathrm{id}_V\, x_1 & B_2 x_0 - \mathrm{id}_V\, x_2 & z x_0 \end{bmatrix}$$

Taking the middle cohomology $\ker\tau/\operatorname{im}\sigma$, we get a framed sheaf (E, Φ) as follows.

The ADHM relation $[B_1, B_2] + zw = 0$ implies $\tau \circ \sigma = 0$, hence we put $E := \ker\tau/\operatorname{im}\sigma$. The stability condition for ADHM data implies that E is torsion-free. Furthermore, if we substitute $x_0 = 0$, then we see that $C^\bullet|_{\ell_\infty}$ is a direct sum of the Koszul resolution and $W \otimes \mathcal{O}_{\ell_\infty}$.

$$C^\bullet|_{\ell_\infty} = 0 \to V \otimes \mathcal{O}_{\ell_\infty}(-1) \xrightarrow{\begin{bmatrix} x_2 \\ -x_1 \\ 0 \end{bmatrix}} \begin{matrix} V \otimes \mathcal{O}_{\ell_\infty} \\ \oplus \\ W \otimes \mathcal{O}_{\ell_\infty} \end{matrix} \xrightarrow{\begin{bmatrix} x_1 & x_2 & 0 \end{bmatrix}} V \otimes \mathcal{O}_{\ell_\infty}(1) \to 0$$

From this, we get a natural framing $\Phi\colon E|_{\ell_\infty} \cong W \otimes \mathcal{O}_{\ell_\infty}$. Then we get a framed sheaf (E, Φ).

For a converse construction

$$(E, \Phi) \mapsto (B_1, B_2, z, w),$$

we refer to [3, Ch. 2]. Hence if we put

$$\begin{aligned} \mathbb{M}(W, V) &= \operatorname{Hom}_{\mathbb{C}}(Q^\vee \otimes V, V) \ni (B_1, B_2) \\ &\oplus \operatorname{Hom}_{\mathbb{C}}(W, V) \ni z \\ &\oplus \operatorname{Hom}_{\mathbb{C}}(\wedge^2 Q^\vee \otimes V, W) \ni w, \\ \mathbb{L}(V) &= \operatorname{Hom}_{\mathbb{C}}(\wedge^2 Q^\vee \otimes V, V) \end{aligned}$$

and

$$\mu\colon \mathbb{M}(W, V) \to \mathbb{L}(V), (B_1, B_2, z, w) \mapsto [B_1, B_2] + zw,$$

then we have the following theorem.

Theorem 4 (Barth)**.** *We have an isomorphism*

$$M(r, n) \cong [\mu^{-1}(0)^{st}/\operatorname{GL}(V)],$$

where $\operatorname{GL}(V)$ *naturally acts on* $\mu^{-1}(0)$*, and* $\mu^{-1}(0)^{st} \subset \mu^{-1}(0)$ *is the stable locus.*

We also consider

$$M_0(r, n) = \operatorname{Spec} \Gamma(\mu^{-1}(0), \mathcal{O}_{\mu^{-1}(0)})^{\operatorname{GL}(V)},$$

and call it instanton moduli, since it is identified with the set of gauge equivalence classes of *framed ASD connections* on $\mathbb{C}^2 = \mathbb{R}^4$. There exists a birational projective morphism

$$\pi\colon M(r, n) \to M_0(r, n).$$

3. Integrations

We consider an algebraic torus

$$T = \mathbb{C}^*,$$

and maximal tori

$$T^2 = \left\{ \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \right\} \subset \mathrm{GL}(Q), T^r = \left\{ \begin{pmatrix} e_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e_r \end{pmatrix} \right\} \subset \mathrm{GL}(W).$$

We have compatible actions

$$\begin{array}{ccc} \mu^{-1}(0) & \curvearrowleft & \mathrm{GL}(V) \times T^2 \times T^r \\ \big\downarrow & & \big\downarrow \\ \mathbb{M}(W, V) & \curvearrowleft & \mathrm{GL}(V) \times \mathrm{GL}(Q) \times \mathrm{GL}(W), \end{array}$$

where $\mathrm{GL}(V) \times \mathrm{GL}(Q) \times \mathrm{GL}(W)$-action on

$$\mathbb{M}(W, V) = \mathrm{Hom}_{\mathbb{C}}(Q^\vee \otimes V, V) \oplus \mathrm{Hom}_{\mathbb{C}}(W, V) \oplus \mathrm{Hom}_{\mathbb{C}}(\wedge^2 Q^\vee \otimes V, W)$$

is naturally defined. This gives $T^2 \times T^r$-action on $M(r,n) = [\mu^{-1}(0)^{st} / \mathrm{GL}(V)]$.

We consider a vector bundle $\mathcal{V} := [\mu^{-1}(0)^{st} \times V / \mathrm{GL}(V)] \to M(r,n) = [\mu^{-1}(0)^{st} / \mathrm{GL}(V)]$ over $M(r,n)$. This bundle $\mathcal{V}$ is called *tautological bundle*, and also have a natural $T^2 \times T^r$-action as above. We introduce another torus $T^{2r} \ni (e^{m_1}, \ldots, e^{m_{2r}})$, and put $\widetilde{T} = T^2 \times T^r \times T^{2r}$.

For integrations, we consider $\widetilde{T}$-equivariant vector bundle

$$\bigoplus_{f=1}^{2r} \mathcal{V} \otimes e^{m_f}$$

over $M(r,n)$, and use the following facts.

- $\pi \colon M(r,n) \to M_0(r,n)$ is $\widetilde{T}$-equivariant.
- $M_0(r,n)^{\widetilde{T}} = \{\mathrm{pt}\}$.
- The $\widetilde{T}$-equivariant Chow ring $A^*_{\widetilde{T}}(\mathrm{pt})$ is equal to $\mathbb{Z}[\boldsymbol{\varepsilon}, \boldsymbol{a}, \boldsymbol{m}]$, where

$$\begin{aligned} \boldsymbol{\varepsilon} &= (c_1(t_1), c_1(t_2)) \\ \boldsymbol{a} &= (c_1(a_1), \ldots, c_1(a_r)) \\ \boldsymbol{m} &= (c_1(m_1), \ldots, c_1(m_{2r})), \end{aligned}$$

  and $t_1, t_2, a_1, \ldots, a_r$ and $m_1, \ldots, m_{2r}$ are identified with weight spaces having corresponding weights $t_1, t_2, a_1, \ldots, a_r$ and $m_1, \ldots, m_{2r}$.

As integrands, we take $\psi := e\left(\bigoplus_{f=1}^{2r} \mathcal{V} \otimes e^{m_f}\right) \in\in A^*_{\widetilde{T}}(M(r,n))$. Since we have $\psi \cap [M(r,n)] \in A_*^{\widetilde{T}}(M(r,n))$, we define integration by

$$\int_{M(r,n)} \psi = (\iota_*)^{-1} \pi_* \left(\psi \cap [M(r,n)] \right) \in \mathbb{Q}(\boldsymbol{\varepsilon}, \boldsymbol{a}, \boldsymbol{m})$$

Here $\iota\colon M_0(r,n)^{\widetilde{T}} \to M_0(r,n)$, and the push-forward $\iota_*\colon A_*^{\widetilde{T}}\left(M_0(r,n)^{\widetilde{T}}\right) \to A_*^{\widetilde{T}}\left(M_0(r,n)\right)$ is an isomorphism after tensoring $\mathbb{Q}(\boldsymbol{\varepsilon},\boldsymbol{a},\boldsymbol{m})$.

We define *Nekrasov partition functions* as

$$Z(\boldsymbol{\varepsilon},\boldsymbol{a},\boldsymbol{m},q) = \sum_{n=0}^{\infty} q^n \int_{M(r,n)} \psi \in \mathbb{Q}(\boldsymbol{\varepsilon},\boldsymbol{a},\boldsymbol{m})[[q]].$$

Our main theorem is the following.

Theorem 5. *We have*

$$Z(-\boldsymbol{\varepsilon},\boldsymbol{a},\boldsymbol{m},q) = (1-(-1)^r)^{u_r} Z(\boldsymbol{\varepsilon},\boldsymbol{a},\boldsymbol{m},q).$$

Here

$$u_r = \frac{(\varepsilon_1+\varepsilon_2)(2\sum_{\alpha=1}^{r} a_\alpha + \sum_{f=1}^{2r})}{\varepsilon_1\varepsilon_2}.$$

In the following, we give an outline of the proof in [5].

Definition 6. ADHM data (B_1,B_2,z,w) are co-stable if $({}^tB_2,{}^tB_1,{}^tw,{}^tz)$ are stable.

We put $M^c(r,n) := \{(B_1,B_2,z,w) \mid ({}^tB_2,{}^tB_1,{}^tw,{}^tz) \in M(r,n)\}$. Then we have

$$Z(-\boldsymbol{\varepsilon},\boldsymbol{a},\boldsymbol{m},q) = \sum_{n=0}^{\infty} q^n \int_{M^c(r,n)} \psi.$$

We analyze wall-crossing phenomena between stability defining $M(r,n)$ and co-stability defining $M^c(r,n)$ by Mochizuki method [2]. This gives our Main thm.

References

[1] Y. Ito, K. Maruyoshi, T. Okuda, *Scheme dependence of instanton counting in ALE spaces*, J. High Energy Phys. 2013, no. 5, 045, front matter+16 pp.

[2] T. Mochizuki, *Donaldson Type Invariants for Algebraic Surfaces: Transition of Moduli Stacks*, Lecture Notes in Math. 1972, Springer, Berlin (2009)

[3] H. Nakajima, *Lectures on Hilbert schemes of points on surfaces*, Univ. Lect. Ser. 18, Amer. Math. Soc., Providence, RI (1999)

[4] H. Nakajima and K. Yoshioka, *Perverse coherent sheaves on blowup. III. Blow-up formula from wall-crossing*, Kyoto Journal of Mathematics 51, No. 2, 263–335 (2011)

[5] R. Ohkawa, *Wall-crossing between stable and co-stable ADHM data*, arXiv:1506.06434.

Research Institute for Mathematical Sciences
Kyoto University
Kyoto 606-8502 JAPAN

E-mail address: ohkawa@kurims.kyoto-u.ac.jp