THE STRUCTURE OF THE SALLY MODULES OF INTEGRALLY CLOSED IDEALS

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Abstract. The purpose of this paper is to give a complete structure theorem of the Sally module of integrally closed ideals $I$ in a Cohen-Macaulay local ring $A$ satisfying the equality $e_1(I) = e_0(I) - \ell_A(A/I) + \ell_A(I^2/QI) + 1$, where $Q$ is a minimal reduction of $I$, and $e_0(I)$ and $e_1(I)$ denote the first two Hilbert coefficients of $I$, respectively. This almost extremal value of $e_1(I)$ with respect to classical inequalities holds a complete description of the homological and the numerical invariants of the associated graded ring.

1. Introduction

This paper is based on a joint work with M. E. Rossi.

Throughout this paper, let $A$ denote a Cohen-Macaulay local ring with maximal ideal $m$ and positive dimension $d$. Let $I$ be an $m$-primary ideal in $A$ and, for simplicity, we assume the residue class field $A/m$ is infinite. Let $e_i(N)$ denote, for an $A$-module $N$, the length of $N$. The integers $\{e_i(I)\}_{0 \leq i \leq d}$ such that the equality

$$\ell_A(A/I^{n+1}) = e_0(I)\binom{n+d}{d} - e_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d e_d(I)$$

holds true for all integers $n \gg 0$, are called the Hilbert coefficients of $A$ with respect to $I$. Choose a parameter ideal $Q$ of $A$ which forms a reduction of $I$ and let $R$ denote the Rees algebra of $A$ and $Q$. Let

$$R = R(I) := A[It] \quad \text{and} \quad T = R(Q) := A[Qt] \subseteq A[t]$$

denote, respectively, the Rees algebras of $I$ and $Q$. Let

$$R' = R'(I) := A[It, t^{-1}] \subseteq A[t, t^{-1}] \quad \text{and} \quad G = G(I) := R'/t^{-1}R' \cong \bigoplus_{n \geq 0} I^n/I^{n+1}.$$ 

Following Vasconcelos [11], we consider

$$S = S_Q(I) = IR/IT \cong \bigoplus_{n \geq 1} I^{n+1}/Q^n I$$

the Sally module of $I$ with respect to $Q$.

The notion of filtration of the Sally module was introduced by M. Vaz Pinto [12] as follows. We denote by $E(\alpha)$, for a graded $T$-module $E$ and each $\alpha \in \mathbb{Z}$, the graded $T$-module whose grading is given by $[E(\alpha)]_n = E_{\alpha+n}$ for all $n \in \mathbb{Z}$.

The detailed version of this paper has been submitted for publication elsewhere.
Definition 1. ([12]) We set, for each \( i \geq 1 \),
\[
C^{(i)} = (I^i R/I^iT)(-i + 1) \cong \bigoplus_{n \geq i} I^{n+1}/Q^{n-i+1} I^i.
\]
and let \( L^{(i)} = [C^{(i)}]_i T \). Then, because \( L^{(i)} \cong \bigoplus_{n \geq i} Q^{n-i}I^{i+1}/Q^{n-i+1}I^i \) and \( C^{(i)}/L^{(i)} \cong C^{(i+1)} \) as graded \( T \)-modules, we have the following natural exact sequences of graded \( T \)-modules
\[
0 \to L^{(i)} \to C^{(i)} \to C^{(i+1)} \to 0
\]
for every \( i \geq 1 \).

We notice that \( C^{(1)} = S \), and \( C^{(i)} \) are finitely generated graded \( T \)-modules for all \( i \geq 1 \), since \( R \) is a module-finite extension of the graded ring \( T \).

So, from now on, we set
\[
C = C_Q(I) = C^{(2)} = (I^2 R/I^2T)(-1)
\]
and we shall explore the structure of \( C \).

Assume that \( I \) is integrally closed. Then, by [1, 3], the inequality
\[
e_1(I) \geq e_0(I) - \ell_A(A/I) + \ell_A(I^2/QI)
\]
holds true and the equality \( e_1(I) = e_0(I) - \ell_A(A/I) + \ell_A(I^2/QI) \) holds if and only if \( I^3 = QI^2 \). When this is the case, the associated graded ring \( G \) of \( I \) is Cohen-Macaulay and the behavior of the Hilbert-Samuel function \( \ell_A(A/I^{n+1}) \) of \( I \) is known (see [1], Corollary 9). Thus the integrally closed ideal \( I \) with \( e_1(I) = e_0(I) - \ell_A(A/I) + \ell_A(I^2/QI) \) enjoys nice properties and it seems natural to ask what happens on the integrally closed ideal \( I \) which satisfies the equality \( e_1(I) = e_0(I) - \ell_A(A/I) + \ell_A(I^2/QI) + 1 \). The problem is not trivial even if we consider \( d = 1 \).

We notice here that \( \ell_A(I^2/QI) = e_0(I) + (d - 1)\ell_A(A/I) - \ell_A(I/I^2) \) holds true (see for instance [9]), so that \( \ell_A(I^2/QI) \) does not depend on a minimal reduction \( Q \) of \( I \).

Let \( B = T/mT \cong (A/m)[X_1, X_2, \ldots, X_d] \) which is a polynomial ring with \( d \) indeterminates over the field \( A/m \). The main result of this paper is stated as follows.

Theorem 2. Assume that \( I \) is integrally closed. Then the following conditions are equivalent:

1. \( e_1(I) = e_0(I) - \ell_A(A/I) + \ell_A(I^2/QI) + 1 \),
2. \( mC = (0) \) and \( \text{rank}_B C = 1 \),
3. \( C \cong (X_1, X_2, \ldots, X_c)B(-1) \) as graded \( T \)-modules for some \( 1 \leq c \leq d \), where \( X_1, X_2, \ldots, X_d \) are linearly independent linear forms of the polynomial ring \( B \).

When this is the case, \( c = \ell_A(I^3/QI^2) \) and \( I^1 = QI^3 \), and the following assertions hold true:

(i) depth \( G \geq d - c \) and \( \text{depth}_T C = d - c + 1 \),
(ii) depth \( G = d - c \), if \( c \geq 2 \).
(iii) Suppose \( c = 1 < d \). Then \( HP_1(n) = \ell_A(A/I^{n+1}) \) for all \( n \geq 0 \) and
the structure of the proof of Theorem 2 in Section 3. In Section 2 we will introduce some auxiliary results on closed ideals and introduce some consequences of Theorem 2. In particular we shall explore the integrally new insights in problems related to the structure of Sally’s module. In Section 4 we will a general setting. Our hope is that these information will be successfully applied to give

The following assertions hold true.

Lemma 3.

(iv) Suppose $2 \leq c < d$. Then $HP_I(n) = \ell_A(A/I^{n+1})$ for all $n \geq 0$ and

$$e_i(I) = \begin{cases} 
  e_1(I) - e_0(I) + \ell_A(A/I) & \text{if } i = 2, \\
  0 & \text{if } i \neq c + 1, c + 2, \ 3 \leq i \leq d \\
  (-1)^{c+1} & \text{if } i = c + 1, c + 2, \ 3 \leq i \leq d 
\end{cases}$$

(v) Suppose $c = d$. Then $HP_I(n) = \ell_A(A/I^{n+1})$ for all $n \geq 2$ and

$$e_i(I) = \begin{cases} 
  e_1(I) - e_0(I) + \ell_A(A/I) & \text{if } i = 2 \text{ and } d \geq 2, \\
  0 & \text{if } 3 \leq i \leq d 
\end{cases}$$

(vi) The Hilbert series $HS_I(z) = \sum_{i \geq 0} \ell_A(I^n/I^{n+1})z^i \in \mathbb{Z}[[t]]$ is given by

$$HS_I(z) = \ell_A(A/I) + \{e_0(I) - \ell_A(A/I) - \ell_A(I^2/QI) - 1\}z + \{\ell_A(I^2/QI) + 1\}z^2 + (1 - z)^{c+1}z.$$  

Let us briefly explain how this paper is organized. We shall introduce an outline of a proof of Theorem 2 in Section 3. In Section 2 we will introduce some auxiliary results on the structure of the $T$-module $C = C_Q(I) = (I^2R/I^2T)(-1)$, some of them are stated in a general setting. Our hope is that these information will be successfully applied to give new insights in problems related to the structure of Sally’s module. In Section 4 we will introduce some consequences of Theorem 2. In particular we shall explore the integrally closed ideals $I$ with $e_1(I) \leq e_0(I) - \ell_A(A/I) + 3$. In Section 5 we will construct a class of Cohen-Macaulay local rings satisfying condition (1) in Theorem 2.

2. Preliminary Steps

The purpose of this section is to summarize some results on the structure of the graded $T$-module $C = C_Q(I) = (I^2R/I^2T)(-1)$, which we need throughout this paper. Remark that in this section $I$ is an $m$-primary ideal not necessarily integrally closed.

Let us begin with the following.

**Lemma 3.** The following assertions hold true.

1. $m^\ell C = (0)$ for integers $\ell \gg 0$; hence $\dim_T C \leq d$.
2. The homogeneous components $\{C_n\}_{n \in \mathbb{Z}}$ of the graded $T$-module $C$ are given by

$$C_n \cong \begin{cases} 
  I^{n+1}/Q^{n-1}I^2 & \text{if } n \leq 1, \\
  (0) & \text{if } n \geq 2. 
\end{cases}$$

3. $C = (0)$ if and only if $I^3 = QI^2$.
4. $mC = (0)$ if and only if $mI^{n+1} \subseteq Q^{n-1}I^2$ for all $n \geq 2$.
5. $S = TC_2$ if and only if $I^4 = QI^3$.

In the following result we need that $Q \cap I^2 = QI$ holds true. This condition is automatically satisfied in the case where $I$ is integrally closed (see [4, 6]).
Lemma 6. Suppose that $Q \cap I^2 = QI$. Then we have $\text{Ass}_T C \subseteq \{mT\}$ so that $\dim_T C = d$, if $C \neq (0)$.

The following Lemma 5 is the crucial fact in the proof of Proposition 4.

Lemma 5. Assume that $Q \cap I^2 = QI$. Then we have $\text{Ass}_T(T/I^2T) = \{mT\}$.

The following techniques are due to M. Vaz Pinto [12, Section 2].

Let $L = L^{(1)} = S_I T$ then $L \cong \bigoplus_{n \geq 1} Q^{n-1}I^2/Q^n I$ and $S/L \cong C$ as graded $T$-modules. Then there exist a canonical exact sequence

$$0 \to L \to S \to C \to 0$$

of graded $T$-modules (Definition 1).

We set $D = (I^2/QI) \otimes_A (T/\text{Ann}_A(I^2/QI))T$. Notice here that $D$ forms a graded $T$-module and $T/\text{Ann}_A(I^2/QI)T \cong (A/\text{Ann}_A(I^2/QI))[X_1, X_2, \ldots, X_d]$ is the polynomial ring with $d$ indeterminates over the ring $A/\text{Ann}_A(I^2/QI)$. Let

$$\theta : D(-1) \to L$$

denotes an epimorphism of graded $T$-modules such that $\theta\left(\sum_{\alpha} x_{\alpha} x_{\alpha}^1 X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_d^{\alpha_d}\right) = \sum_{\alpha} x_{\alpha} x_{\alpha}^1 a_{\alpha}^2 \cdots a_{d}^2 \alpha_d t^{\alpha_d+1}$ for $x_{\alpha} \in I^2$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{Z}^d$ with $\alpha_i \geq 0$ ($1 \leq i \leq d$), where $|\alpha| = \sum_{i=1}^d \alpha_i$, and $x_{\alpha}$ and $x_{\alpha} x_{\alpha}^1 a_{\alpha}^2 \cdots a_{d}^2 \alpha_d t^{\alpha_d+1}$ denote the images of $x_{\alpha}$ in $I^2/QI$ and $x_{\alpha} x_{\alpha}^1 a_{\alpha}^2 \cdots a_{d}^2 \alpha_d t^{\alpha_d+1}$ in $L$.

Then we have the following lemma.

Lemma 6. Suppose that $Q \cap I^2 = QI$. Then the map $\theta : D(-1) \to L$ is an isomorphism of graded $T$-modules.

Thanks to Lemma 6 and [2, Proposition 2.2 (2)], we can prove the following result.

Proposition 7. Suppose that $Q \cap I^2 = QI$. Then we have

$$\ell_A(A/I^{n+1}) = e_0(I)\binom{n+d}{d} - \{e_0(I) - \ell_A(A/I) + \ell_A(I^2/QI)\} \binom{n+d-1}{d-1}$$

$$+ \ell_A(I^2/QI) \binom{n+d-2}{d-2} - \ell_A(C_\alpha)$$

for all $n \geq 0$.

The following result specifies [2, Proposition 2.2 (3)] and, by using Proposition 4 and 7, the proof takes advantage of the same techniques.

Proposition 8. Suppose that $Q \cap I^2 = QI$. Let $p = mT$. Then we have

$$e_1(I) = e_0(I) - \ell_A(A/I) + \ell_A(I^2/QI) + \ell_T(C_p).$$

Combining Lemma 3 (3) and Proposition 8 we obtain the following result that was proven by Elias and Valla [1, Theorem 2.1] in the case where $I = m$.

Corollary 9. Suppose that $Q \cap I^2 = QI$. Then we have $e_1(I) \geq e_0(I) - \ell_A(A/I) + \ell_A(I^2/QI)$. The equality $e_1(I) = e_0(I) - \ell_A(A/I) + \ell_A(I^2/QI)$ holds true if and only if $I^3 = QI^2$. When this is the case, $e_0(I) = e_1(I) - e_0(I) + \ell_A(A/I)$ if $d \geq 2$, $e_i(I) = 0$ for all $3 \leq i \leq d$, and $G$ is a Cohen-Macaulay ring.
In the end of this section, let us introduce the relationship between the depth of the module $C$ and the associated graded ring $G$ of $I$.

**Lemma 10.** Suppose that $Q \cap I^2 = QI$ and $C \neq (0)$. Let $s = \text{depth}_T C$. Then we have $\text{depth} G \geq s - 1$. In particular, we have $\text{depth} G = s - 1$, if $s \leq d - 2$.

### 3. Outline of proof of Theorem 2

The purpose of this section is to prove Theorem 2. Throughout this section, let $I$ be an integrally closed $m$-primary ideal. The following theorem is the key.

**Theorem 11.** Suppose that $I$ is integrally closed. Then the following conditions are equivalent:

1. $e_1(I) = e_0(I) - \ell_A(A/I) + \ell_A(I^2/QI) + 1$,
2. $mC = (0)$ and $\text{rank}_B C = 1$,
3. there exists a non-zero graded ideal $a$ of $B$ such that $C \cong a(-1)$ as graded $T$-modules.

To prove Theorem 11, we need the following bound on $e_2(I)$.

**Lemma 12.** ([5, Theorem 12], [10, Corollary 2.5], [9, Corollary 3.1], ) Suppose $d \geq 2$ and let $I$ be an integrally closed ideal, then $e_2(I) \geq e_1(I) - e_0(I) + \ell_A(A/I)$.

As a direct consequence of Theorem 11 the following result holds true.

**Proposition 13.** Assume that $I$ is integrally closed. Suppose that $e_1(I) = e_0(I) - \ell_A(A/I) + \ell_A(I^2/QI) + 1$ and $I^4 = QI^3$ and let $c = \ell_A(I^3/QI^2)$. Then

1. $1 \leq c \leq d$ and $\mu_B(C) = c$.
2. $\text{depth} G \geq d - c$ and $\text{depth}_T C = d - c + 1$,
3. $\text{depth} G = d - c$, if $c \geq 2$.
4. Suppose $c = 1 < d$. Then $HP_1(n) = \ell_A(A/I^{n+1})$ for all $n \geq 0$ and

$$e_i(I) = \begin{cases} e_1(I) - e_0(I) + \ell_A(A/I) + \ell_A(I^{n+1}) & \text{if } i = 2, \\ 1 & \text{if } i = 3 \text{ and } d \geq 3, \\ 0 & \text{if } 4 \leq i \leq d. \end{cases}$$

5. Suppose $2 \leq c < d$. Then $HP_1(n) = \ell_A(A/I^{n+1})$ for all $n \geq 0$ and

$$e_i(I) = \begin{cases} e_1(I) - e_0(I) + \ell_A(A/I) & \text{if } i = 2, \\ 0 & \text{if } i \neq c + 1, c + 2, \ 3 \leq i \leq d \\ (-1)^{c+1} & \text{if } i = c + 1, c + 2, \ 3 \leq i \leq d \end{cases}$$

6. Suppose $c = d$. Then $HP_1(n) = \ell_A(A/I^{n+1})$ for all $n \geq 2$ and

$$e_i(I) = \begin{cases} e_1(I) - e_0(I) + \ell_A(A/I) & \text{if } i = 2 \text{ and } d \geq 2, \\ 0 & \text{if } 3 \leq i \leq d \end{cases}$$

7. The Hilbert series $HS_I(z)$ is given by

$$HS_I(z) = \frac{\ell_A(A/I) + \{e_0(I) - \ell_A(A/I) - \ell_A(I^2/QI) - 1\}z + \{\ell_A(I^2/QI) + 1\}z^2 + (1 - z)^{c+1}z}{(1 - z)^d}.$$
We prove now Theorem 2. Assume assertion (1) in Theorem 2. Then we have an isomorphism \( C \cong a(-1) \) as graded \( B \)-modules for a graded ideal \( a \) in \( B \) by Theorem 11. Once we are able to show \( I^4 = QI^3 \), then, because \( C = TC_2 \) by Lemma 3 (5), the ideal \( a \) is generated by linearly independent linear forms \( \{X_{i}\}_{1 \leq i \leq c} \) of \( B \) with \( c = \ell_A(I^3/QI^2) \) (recall that \( a_1 \cong C_2 \cong I^3/QI^2 \) by Lemma 3 (2)). Therefore, the implication \( (1) \Rightarrow (3) \) in Theorem 2 follows. We also notice that, the last assertions of Theorem 2 follow by Proposition 13.

Thus our Theorem 2 has been proven modulo the following theorem.

**Theorem 14.** Assume that \( I \) is integrally closed. Suppose that \( e_1(I) = e_0(I) - \ell_A(A/I) + \ell_A(I^2/QI) + 1 \). Then \( I^4 = QI^3 \).

### 4. Consequences

The purpose of this section is to present some consequences of Theorem 2.

We explore the relationship between the inequality of Northcott [7] and the structure of the graded module \( C \) of an integrally closed ideal.

It is well known that, for an \( m \)-primary ideal \( I \) in a Cohen-Macaulay local ring \((A,m)\), the inequality

\[
e_1(I) \geq e_0(I) - \ell_A(A/I)
\]

holds true ([7]) and the equality holds if and only if \( I^2 = QI \) ([4, Theorem 2.1]). When this is the case, the associated graded ring \( G \) of \( I \) is Cohen-Macaulay.

Suppose that \( I \) is integrally closed and \( e_1(I) = e_0(I) - \ell_A(A/I) + 1 \) then, thanks to [5, Corollary 14], we have \( I^3 = QI^2 \) and the associated graded ring \( G \) of \( I \) is Cohen-Macaulay. Thus the integrally closed ideal \( I \) with \( e_1(I) \leq e_0(I) - \ell_A(A/I) + 1 \) seems satisfactory understood. In this section, we briefly study the integrally closed ideals \( I \) with \( e_1(I) = e_0(I) - \ell_A(A/I) + 2 \), and \( e_1(I) = e_0(I) - \ell_A(A/I) + 3 \).

Let us begin with the following.

**Theorem 15.** Assume that \( I \) is integrally closed. Suppose that \( e_1(I) = e_0(I) - \ell_A(A/I) + 2 \) and \( I^3 \neq QI^2 \). Then the following assertions hold true.

1. \( \ell_A(I^2/QI) = \ell_A(I^3/QI^2) = 1 \), and \( I^4 = QI^3 \).
2. \( C \cong B(-2) \) as graded \( T \)-modules.
3. Depth \( G = d - 1 \).
4. \( e_2(I) = 3 \) if \( d \geq 2 \), \( e_3(I) = 1 \) if \( d \geq 3 \), and \( e_i(I) = 0 \) for \( 4 \leq i \leq d \).
5. The Hilbert series \( HS_I(z) \) is given by

\[
HS_I(z) = \frac{\ell_A(A/I) + \{e_0(I) - \ell_A(A/I) - 1\}z + z^3}{(1 - z)^d}.
\]

Notice that the following result also follows by [9, Theorem 4.6].

**Corollary 16.** Assume that \( I \) is integrally closed and suppose that \( e_1(I) = e_0(I) - \ell_A(A/I) + 2 \). Then depth \( G \geq d - 1 \) and \( I^4 = QI^3 \), and the graded ring \( G \) is Cohen-Macaulay if and only if \( I^3 = QI^2 \).
Before closing this section, we briefly study the integrally closed ideal $I$ with $e_1(I) = e_0(I) - \ell_A(A/I) + 3$. Suppose that $e_1(I) = e_0(I) - \ell_A(A/I) + 3$ then we have
\[ 0 < \ell_A(I^2/QI) \leq c_1(I) - e_0(I) + \ell_A(A/I) = 3 \]
by Corollary 9. If $\ell_A(I^2/QI) = 1$ then we have $\operatorname{depth} G \geq d - 1$ by [8, 13]. If $\ell_A(I^2/QI) = 3$ then the equality $e_1(I) = e_0(I) - \ell_A(A/I) + \ell_A(I^2/QI)$ holds true, so that $I^3 = QI^2$ and the associated graded ring $G$ of $I$ is Cohen-Macaulay by Corollary 9. Thus we need to consider the following.

**Theorem 17.** Suppose that $d \geq 2$. Assume that $I$ is integrally closed and $e_1(I) = e_0(I) - \ell_A(A/I) + 3$ and $\ell_A(I^2/QI) = 2$. Let $c = \ell_A(I^3/QI^2)$. Then the following assertions hold true.

1. Either $C \cong B(-2)$ as graded $T$-modules or there exists an exact sequence
\[ 0 \to B(-3) \to B(-2) \oplus B(-2) \to C \to 0 \]
of graded $T$-modules.
2. $1 \leq c \leq 2$ and $I^3 = QI^3$.
3. Suppose $c = 1$ then $\operatorname{depth} G \geq d - 1$ and $e_2(I) = 4$, $e_3(I) = 1$ if $d \geq 3$, and $e_i(I) = 0$ for $4 \leq i \leq d$.
4. Suppose $c = 2$ then $\operatorname{depth} G = d - 2$ and $e_2(I) = 3$, $e_3(I) = -1$ if $d \geq 3$, $e_4(I) = -1$ if $d \geq 4$, and $e_i(I) = 0$ for $5 \leq i \leq d$.
5. The Hilbert series $HS_I(z)$ is given by
\[
HS_I(z) = \begin{cases} 
\frac{\ell_A(A/I) + \{e_0(I) - \ell_A(A/I) - 2\}z + z^2 + z^3}{(1 - z)^d}, & \text{if } c = 1, \\
\frac{\ell_A(A/I) + \{e_0(I) - \ell_A(A/I) - 2\}z + 3z^3 - z^4}{(1 - z)^d}, & \text{if } c = 2.
\end{cases}
\]

We remark that $\ell_A(I^2/QI)$ measures how far is the multiplicity of $I$ from the minimal value, see [9, Corollary 2.1]. If $\ell_A(I^2/QI) \leq 1$, then depth $G \geq d - 1$, but it is still open the problem whether $\operatorname{depth} G \geq d - 2$, assuming $\ell_A(I^2/QI) = 2$. Theorem 17 confirms the conjectured bound.

**Corollary 18.** Assume that $I$ is integrally closed. Suppose that $e_1(I) = e_0(I) - \ell_A(A/I) + 3$. Then $\operatorname{depth} G \geq d - 2$.

5. **An Example**

The goal of this section is to construct an example of Cohen-Macaulay local ring with the maximal ideal $m$ satisfying the equality in Theorem 2 (1). The class of examples we exhibit includes an interesting example given by H.-J. Wang, see [9, Example 3.2].

**Theorem 19.** Let $d \geq c \geq 1$ be integers. Then there exists a Cohen-Macaulay local ring $(A, m)$ such that
\[ d = \dim A, \quad e_1(m) = e_0(m) + \ell_A(m^2/Qm), \quad \text{and } c = \ell_A(m^3/Qm^2) \]
for some minimal reduction $Q = (a_1, a_2, \ldots, a_d)$ of $m$. 

\[ -7 - \]
To construct necessary examples we may assume that \( c = d \).

Let \( m \geq 0 \) and \( d \geq 1 \) be integers. Let

\[
D = k[[\{X_j\}_{1 \leq j \leq m}, Y; \{V_i\}_{1 \leq i \leq d}, \{Z_i\}_{1 \leq i \leq d}]]
\]

be the formal power series ring with \( m + 2d + 1 \) indeterminates over an infinite field \( k \), and let

\[
a = [(X_j | 1 \leq j \leq m) + (Y)] : [(X_j | 1 \leq j \leq m) + (Y) + (V_i | 1 \leq i \leq d)]
\]

\[
+ (V_iV_j | 1 \leq i, j \leq d, i \neq j) + (V_i^3 - Z_iY | 1 \leq i \leq d).
\]

We set \( A = D/a \) and denote the images of \( X_j, Y, V_i, \) and \( Z_i \) in \( A \) by \( x_j, y, v_i, \) and \( a_i \), respectively. Then, since \( \sqrt{a} = (X_j | 1 \leq j \leq m) + (Y) + (V_i | 1 \leq i \leq d) \), we have \( \dim A = d \). Let \( m = (x_j | 1 \leq j \leq m) + (y) + (v_i | 1 \leq i \leq d) + (a_i | 1 \leq i \leq d) \) be the maximal ideal in \( A \) and we set \( Q = (a_i | 1 \leq i \leq d) \). Then, \( m^2 = Qm + (v_i^3 | 1 \leq i \leq d), m^3 = Qm^2 + Qy, \) and \( m^4 = Qm^3 \). Therefore \( Q \) is a minimal reduction of \( m \), and \( a_1, a_2, \ldots, a_d \) is a system of parameters for \( A \). We then have the following.

**Theorem 20.** The following assertions hold true.

1. \( A \) is a Cohen-Macaulay local ring with \( \dim A = d \).
2. \( C_Q(m) \cong B_d(-1) \) as graded \( T \)-modules. Therefore, \( \ell_A(m^3/Qm^2) = d \).
3. \( e_0(m) = m + 2d + 2, \ e_1(m) = m + 3d + 2 \).
4. \( e_2(m) = d + 1 \) if \( d \geq 2 \), and \( e_i(m) = 0 \) for all \( 3 \leq i \leq d \).
5. \( G(m) \) is a Buchsbaum ring with \( \text{depth} G(m) = 0 \) and \( \mathbb{I}(G(m)) = d \).

**References**


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