THE STRUCTURE OF THE SALLY MODULES OF INTEGRALLY CLOSED IDEALS

KAZUHO OZEKI

Abstract. The purpose of this paper is to give a complete structure theorem of the Sally module of integrally closed ideals I in a Cohen-Macaulay local ring A satisfying the equality $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI) + 1$, where Q is a minimal reduction of I, and $\mathrm{e}_0(I)$ and $\mathrm{e}_1(I)$ denote the first two Hilbert coefficients of I, respectively. This almost extremal value of $\mathrm{e}_1(I)$ with respect to classical inequalities holds a complete description of the homological and the numerical invariants of the associated graded ring.

1. Introduction

This paper is based on a joint work with M. E. Rossi.

Throughout this paper, let A denote a Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$ and positive dimension d. Let I be an $\mathfrak{m}$-primary ideal in A and, for simplicity, we assume the residue class field $A/\mathfrak{m}$ is infinite. Let $\ell_A(N)$ denote, for an A-module N, the length of N. The integers $\{\mathrm{e}_i(I)\}_{0 \le i \le d}$ such that the equality

$$\ell_A(A/I^{n+1}) = \mathrm{e}_0(I)\binom{n+d}{d} - \mathrm{e}_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d \mathrm{e}_d(I)$$

holds true for all integers $n \gg 0$, are called the *Hilbert coefficients* of A with respect to I. Choose a parameter ideal Q of A which forms a reduction of I and let

$$R = \mathrm{R}(I) := A[It] \quad \text{and} \quad T = \mathrm{R}(Q) := A[Qt] \quad \subseteq A[t]$$

denote, respectively, the Rees algebras of I and Q. Let

$$R' = \mathrm{R}'(I) := A[It, t^{-1}] \ \subseteq A[t, t^{-1}] \quad \text{and} \quad G = \mathrm{G}(I) := R'/t^{-1}R' \cong \bigoplus_{n \ge 0} I^n/I^{n+1}.$$

Following Vasconcelos [11], we consider

$$S = \mathrm{S}_Q(I) = IR/IT \cong \bigoplus_{n \ge 1} I^{n+1}/Q^nI$$

the Sally module of I with respect to Q.

The notion of *filtration of the Sally module* was introduced by M. Vaz Pinto [12] as follows. We denote by $E(\alpha)$, for a graded T-module E and each $\alpha \in \mathbb{Z}$, the graded T-module whose grading is given by $[E(\alpha)]_n = E_{\alpha+n}$ for all $n \in \mathbb{Z}$.

The detailed version of this paper has been submitted for publication elsewhere.

Definition 1. ([12]) We set, for each $i \geq 1$,

$$C^{(i)} = (I^iR/I^iT)(-i+1) \cong \bigoplus_{n \geq i} I^{n+1}/Q^{n-i+1}I^i.$$

and let $L^{(i)} = [C^{(i)}]_i\ T$. Then, because $L^{(i)} \cong \bigoplus_{n\geq i} Q^{n-i}I^{i+1}/Q^{n-i+1}I^i$ and $C^{(i)}/L^{(i)} \cong C^{(i+1)}$ as graded T-modules, we have the following natural exact sequences of graded T-modules

$$0 \to L^{(i)} \to C^{(i)} \to C^{(i+1)} \to 0$$

for every $i \geq 1$.

We notice that $C^{(1)} = S$, and $C^{(i)}$ are finitely generated graded T-modules for all $i \geq 1$, since R is a module-finite extension of the graded ring T.

So, from now on, we set

$$C = \mathrm{C}_Q(I) = C^{(2)} = (I^2R/I^2T)(-1)$$

and we shall explore the structure of C.

Assume that I is integrally closed. Then, by [1, 3], the inequality

$$\mathrm{e}_1(I) \geq \mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI)$$

holds true and the equality $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI)$ holds if and only if $I^3 = QI^2$. When this is the case, the associated graded ring G of I is Cohen-Macaulay and the behavior of the Hilbert-Samuel function $\ell_A(A/I^{n+1})$ of I is known (see [1], Corollary 9). Thus the integrally closed ideal I with $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI)$ enjoys nice properties and it seems natural to ask what happens on the integrally closed ideal I which satisfies the equality $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI) + 1$. The problem is not trivial even if we consider $d = 1$.

We notice here that $\ell_A(I^2/QI) = \mathrm{e}_0(I) + (d-1)\ell_A(A/I) - \ell_A(I/I^2)$ holds true (see for instance [9]), so that $\ell_A(I^2/QI)$ does not depend on a minimal reduction Q of I.

Let $B = T/\mathfrak{m}T \cong (A/\mathfrak{m})[X_1, X_2, \cdots, X_d]$ which is a polynomial ring with d indeterminates over the field $A/\mathfrak{m}$. The main result of this paper is stated as follows.

Theorem 2. *Assume that I is integrally closed. Then the following conditions are equivalent:*

(1) $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI) + 1$,
(2) $\mathfrak{m}C = (0)$ *and* $\mathrm{rank}_B\, C = 1$,
(3) $C \cong (X_1, X_2, \cdots, X_c)B(-1)$ *as graded T-modules for some $1 \leq c \leq d$, where $X_1, X_2, \cdots, X_d$ are linearly independent linear forms of the polynomial ring B.*

When this is the case, $c = \ell_A(I^3/QI^2)$ and $I^4 = QI^3$, and the following assertions hold true:

(i) depth $G \geq d - c$ *and* $\mathrm{depth}_T\, C = d - c + 1$,
(ii) depth $G = d - c$, *if $c \geq 2$.*
(iii) *Suppose $c = 1 < d$. Then $HP_I(n) = \ell_A(A/I^{n+1})$ for all $n \geq 0$ and*

$$\mathrm{e}_i(I) = \begin{cases} \mathrm{e}_1(I) - \mathrm{e}_0(I) + \ell_A(A/I) + 1 & \textit{if } i = 2, \\ 1 & \textit{if } i = 3 \textit{ and } d \geq 3, \\ 0 & \textit{if } 4 \leq i \leq d. \end{cases}$$

(iv) *Suppose* $2 \leq c < d$. *Then* $HP_I(n) = \ell_A(A/I^{n+1})$ *for all* $n \geq 0$ *and*

$$\mathrm{e}_i(I) = \begin{cases} \mathrm{e}_1(I) - \mathrm{e}_0(I) + \ell_A(A/I) & \textit{if } i = 2, \\ 0 & \textit{if } i \neq c+1, c+2, \ \ 3 \leq i \leq d \\ (-1)^{c+1} & \textit{if } i = c+1, c+2, \ \ 3 \leq i \leq d \end{cases}$$

(v) *Suppose* $c = d$. *Then* $HP_I(n) = \ell_A(A/I^{n+1})$ *for all* $n \geq 2$ *and*

$$\mathrm{e}_i(I) = \begin{cases} \mathrm{e}_1(I) - \mathrm{e}_0(I) + \ell_A(A/I) & \textit{if } i = 2 \textit{ and } d \geq 2, \\ 0 & \textit{if } 3 \leq i \leq d \end{cases}$$

(vi) *The Hilbert series* $HS_I(z) = \sum_{t \geq 0} \ell_A(I^n/I^{n+1}) z^t \in \mathbb{Z}[[t]]$ *is given by*

$$HS_I(z) = \frac{\ell_A(A/I) + \{\mathrm{e}_0(I) - \ell_A(A/I) - \ell_A(I^2/QI) - 1\}z + \{\ell_A(I^2/QI) + 1\}z^2 + (1-z)^{c+1}z}{(1-z)^d}.$$

Let us briefly explain how this paper is organized. We shall introduce an outline of a proof of Theorem 2 in Section 3. In Section 2 we will introduce some auxiliary results on the structure of the T-module $C = \mathrm{C}_Q(I) = (I^2R/I^2T)(-1)$, some of them are stated in a general setting. Our hope is that these information will be successfully applied to give new insights in problems related to the structure of Sally's module. In Section 4 we will introduce some consequences of Theorem 2. In particular we shall explore the integrally closed ideals I with $\mathrm{e}_1(I) \leq \mathrm{e}_0(I) - \ell_A(A/I) + 3$. In Section 5 we will construct a class of Cohen-Macaulay local rings satisfying condition (1) in Theorem 2.

2. Preliminary Steps

The purpose of this section is to summarize some results on the structure of the graded T-module $C = \mathrm{C}_Q(I) = (I^2R/I^2T)(-1)$, which we need throughout this paper. Remark that in this section I is an $\mathfrak{m}$-primary ideal not necessarily integrally closed.

Let us begin with the following.

Lemma 3. *The following assertions hold true.*

(1) $\mathfrak{m}^\ell C = (0)$ *for integers* $\ell \gg 0$; *hence* $\dim_T C \leq d$.

(2) *The homogeneous components* $\{C_n\}_{n \in \mathbb{Z}}$ *of the graded* T*-module* C *are given by*

$$C_n \cong \begin{cases} (0) & \textit{if } n \leq 1, \\ I^{n+1}/Q^{n-1}I^2 & \textit{if } n \geq 2. \end{cases}$$

(3) $C = (0)$ *if and only if* $I^3 = QI^2$.

(4) $\mathfrak{m}C = (0)$ *if and only if* $\mathfrak{m}I^{n+1} \subseteq Q^{n-1}I^2$ *for all* $n \geq 2$.

(5) $S = TC_2$ *if and only if* $I^4 = QI^3$.

In the following result we need that $Q \cap I^2 = QI$ holds true. This condition is automatically satisfied in the case where I is integrally closed (see [4, 6]).

Proposition 4. *Suppose that $Q \cap I^2 = QI$. Then we have $\operatorname{Ass}_T C \subseteq \{\mathfrak{m}T\}$ so that $\dim_T C = d$, if $C \neq (0)$.*

The following Lemma 5 is the crucial fact in the proof of Proposition 4.

Lemma 5. *Assume that $Q \cap I^2 = QI$. Then we have $\operatorname{Ass}_T(T/I^2T) = \{\mathfrak{m}T\}$.*

The following techniques are due to M. Vaz Pinto [12, Section 2].

Let $L = L^{(1)} = S_1T$ then $L \cong \bigoplus_{n \geq 1} Q^{n-1}I^2/Q^nI$ and $S/L \cong C$ as graded T-modules. Then there exist a canonical exact sequence

$$0 \to L \to S \to C \to 0 \quad (\dagger)$$

of graded T-modules (Definition 1).

We set $D = (I^2/QI) \otimes_A (T/\operatorname{Ann}_A(I^2/QI)T)$. Notice here that D forms a graded T-module and $T/\operatorname{Ann}_A(I^2/QI)T \cong (A/\operatorname{Ann}_A(I^2/QI))[X_1, X_2, \cdots, X_d]$ is the polynomial ring with d indeterminates over the ring $A/\operatorname{Ann}_A(I^2/QI)$. Let

$$\theta : D(-1) \to L$$

denotes an epimorphism of graded T-modules such that $\theta(\sum_\alpha \overline{x_\alpha} \otimes X_1^{\alpha_1}X_2^{\alpha_2}\cdots X_d^{\alpha_d}) = \sum_\alpha \overline{x_\alpha a_1^{\alpha_1}a_2^{\alpha_2}\cdots a_d^{\alpha_d}t^{|\alpha|+1}}$ for $x_\alpha \in I^2$ and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d) \in \mathbb{Z}^d$ with $\alpha_i \geq 0$ $(1 \leq i \leq d)$, where $|\alpha| = \sum_{i=1}^d \alpha_i$, and $\overline{x_\alpha}$ and $\overline{x_\alpha a_1^{\alpha_1}a_2^{\alpha_2}\cdots a_d^{\alpha_d}t^{|\alpha|+1}}$ denote the images of x_α in I^2/QI and $x_\alpha a_1^{\alpha_1}a_2^{\alpha_2}\cdots a_d^{\alpha_d}t^{|\alpha|+1}$ in L.

Then we have the following lemma.

Lemma 6. *Suppose that $Q \cap I^2 = QI$. Then the map $\theta : D(-1) \to L$ is an isomorphism of graded T-modules.*

Thanks to Lemma 6 and [2, Proposition 2.2 (2)], we can prove the following result.

Proposition 7. *Suppose that $Q \cap I^2 = QI$. Then we have*

$$\begin{aligned}\ell_A(A/I^{n+1}) &= \mathrm{e}_0(I)\binom{n+d}{d} - \{\mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI)\}\binom{n+d-1}{d-1} \\ &\quad + \ell_A(I^2/QI)\binom{n+d-2}{d-2} - \ell_A(C_n)\end{aligned}$$

for all $n \geq 0$.

The following result specifies [2, Proposition 2.2 (3)] and, by using Proposition 4 and 7, the proof takes advantage of the same techniques.

Proposition 8. *Suppose that $Q \cap I^2 = QI$. Let $\mathfrak{p} = \mathfrak{m}T$. Then we have*

$$\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI) + \ell_{T_\mathfrak{p}}(C_\mathfrak{p}).$$

Combining Lemma 3 (3) and Proposition 8 we obtain the following result that was proven by Elias and Valla [1, Theorem 2.1] in the case where $I = \mathfrak{m}$.

Corollary 9. *Suppose that $Q \cap I^2 = QI$. Then we have $\mathrm{e}_1(I) \geq \mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI)$. The equality $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI)$ holds true if and only if $I^3 = QI^2$. When this is the case, $\mathrm{e}_2(I) = \mathrm{e}_1(I) - \mathrm{e}_0(I) + \ell_A(A/I)$ if $d \geq 2$, $\mathrm{e}_i(I) = 0$ for all $3 \leq i \leq d$, and G is a Cohen-Macaulay ring.*

In the end of this section, let us introduce the relationship between the depth of the module C and the associated graded ring G of I.

Lemma 10. *Suppose that* $Q \cap I^2 = QI$ *and* $C \neq (0)$. *Let* $s = \mathrm{depth}_T C$. *Then we have* $\mathrm{depth} G \geq s-1$. *In particular, we have* $\mathrm{depth} G = s-1$, *if* $s \leq d-2$.

3. Outline of proof of Theorem 2

The purpose of this section is to prove Theorem 2. Throughout this section, let I be an integrally closed $\mathfrak{m}$-primary ideal. The following theorem is the key.

Theorem 11. *Suppose that* I *is integrally closed. Then the following conditions are equivalent:*

(1) $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI) + 1$,
(2) $\mathfrak{m}C = (0)$ *and* $\mathrm{rank}_B\, C = 1$,
(3) *there exists a non-zero graded ideal* $\mathfrak{a}$ *of* B *such that* $C \cong \mathfrak{a}(-1)$ *as graded* T-*modules.*

To prove Theorem 11, we need the following bound on $\mathrm{e}_2(I)$.

Lemma 12. ([5, Theorem 12], [10, Corollary 2.5], [9, Corollary 3.1],) *Suppose* $d \geq 2$ *and let* I *be an integrally closed ideal, then* $\mathrm{e}_2(I) \geq \mathrm{e}_1(I) - \mathrm{e}_0(I) + \ell_A(A/I)$.

As a direct consequence of Theorem 11 the following result holds true.

Proposition 13. *Assume that* I *is integrally closed. Suppose that* $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI) + 1$ *and* $I^4 = QI^3$ *and let* $c = \ell_A(I^3/QI^2)$. *Then*

(1) $1 \leq c \leq d$ *and* $\mu_B(C) = c$.
(2) $\mathrm{depth}\, G \geq d-c$ *and* $\mathrm{depth}_T C = d-c+1$,
(3) $\mathrm{depth}\, G = d-c$, *if* $c \geq 2$.
(4) *Suppose* $c = 1 < d$. *Then* $HP_I(n) = \ell_A(A/I^{n+1})$ *for all* $n \geq 0$ *and*

$$\mathrm{e}_i(I) = \begin{cases} \mathrm{e}_1(I) - \mathrm{e}_0(I) + \ell_A(A/I) + 1 & \text{if } i = 2, \\ 1 & \text{if } i = 3 \text{ and } d \geq 3, \\ 0 & \text{if } 4 \leq i \leq d. \end{cases}$$

(5) *Suppose* $2 \leq c < d$. *Then* $HP_I(n) = \ell_A(A/I^{n+1})$ *for all* $n \geq 0$ *and*

$$\mathrm{e}_i(I) = \begin{cases} \mathrm{e}_1(I) - \mathrm{e}_0(I) + \ell_A(A/I) & \text{if } i = 2, \\ 0 & \text{if } i \neq c+1, c+2,\ \ 3 \leq i \leq d \\ (-1)^{c+1} & \text{if } i = c+1, c+2,\ \ 3 \leq i \leq d \end{cases}$$

(6) *Suppose* $c = d$. *Then* $HP_I(n) = \ell_A(A/I^{n+1})$ *for all* $n \geq 2$ *and*

$$\mathrm{e}_i(I) = \begin{cases} \mathrm{e}_1(I) - \mathrm{e}_0(I) + \ell_A(A/I) & \text{if } i = 2 \text{ and } d \geq 2, \\ 0 & \text{if } 3 \leq i \leq d \end{cases}$$

(7) *The Hilbert series* $HS_I(z)$ *is given by*

$$HS_I(z) = \frac{\ell_A(A/I) + \{\mathrm{e}_0(I) - \ell_A(A/I) - \ell_A(I^2/QI) - 1\}z + \{\ell_A(I^2/QI) + 1\}z^2 + (1-z)^{c+1}z}{(1-z)^d}.$$

We prove now Theorem 2. Assume assertion (1) in Theorem 2. Then we have an isomorphism $C \cong \mathfrak{a}(-1)$ as graded B-modules for a graded ideal $\mathfrak{a}$ in B by Theorem 11. Once we are able to show $I^4 = QI^3$, then, because $C = TC_2$ by Lemma 3 (5), the ideal $\mathfrak{a}$ is generated by linearly independent linear forms $\{X_i\}_{1 \le i \le c}$ of B with $c = \ell_A(I^3/QI^2)$ (recall that $\mathfrak{a}_1 \cong C_2 \cong I^3/QI^2$ by Lemma 3 (2)). Therefore, the implication (1) $\Rightarrow$ (3) in Theorem 2 follows. We also notice that, the last assertions of Theorem 2 follow by Proposition 13.

Thus our Theorem 2 has been proven modulo the following theorem.

Theorem 14. *Assume that I is integrally closed. Suppose that* $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI) + 1$*. Then $I^4 = QI^3$.*

4. Consequences

The purpose of this section is to present some consequences of Theorem 2.

We explore the relationship between the inequality of Northcott [7] and the structure of the graded module C of an integrally closed ideal.

It is well known that, for an $\mathfrak{m}$-primary ideal I in a Cohen-Macaulay local ring $(A, \mathfrak{m})$, the inequality

$$\mathrm{e}_1(I) \ge \mathrm{e}_0(I) - \ell_A(A/I)$$

holds true ([7]) and the equality holds if and only if $I^2 = QI$ ([4, Theorem 2.1]). When this is the case, the associated graded ring G of I is Cohen-Macaulay.

Suppose that I is integrally closed and $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 1$ then, thanks to [5, Corollary 14], we have $I^3 = QI^2$ and the associated graded ring G of I is Cohen-Macaulay. Thus the integrally closed ideal I with $\mathrm{e}_1(I) \le \mathrm{e}_0(I) - \ell_A(A/I) + 1$ seems satisfactory understood. In this section, we briefly study the integrally closed ideals I with $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 2$, and $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 3$.

Let us begin with the following.

Theorem 15. *Assume that I is integrally closed. Suppose that* $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 2$ *and $I^3 \ne QI^2$. Then the following assertions hold true.*

(1) $\ell_A(I^2/QI) = \ell_A(I^3/QI^2) = 1$, *and* $I^4 = QI^3$.
(2) $C \cong B(-2)$ *as graded T-modules.*
(3) $\operatorname{depth} G = d - 1$.
(4) $\mathrm{e}_2(I) = 3$ *if* $d \ge 2$, $\mathrm{e}_3(I) = 1$ *if* $d \ge 3$, *and* $\mathrm{e}_i(I) = 0$ *for* $4 \le i \le d$.
(5) *The Hilbert series $HS_I(z)$ is given by*

$$HS_I(z) = \frac{\ell_A(A/I) + \{\mathrm{e}_0(I) - \ell_A(A/I) - 1\}z + z^3}{(1-z)^d}.$$

Notice that the following result also follows by [9, Theorem 4.6].

Corollary 16. *Assume that I is integrally closed and suppose that* $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 2$*. Then* $\operatorname{depth} G \ge d - 1$ *and $I^4 = QI^3$, and the graded ring G is Cohen-Macaulay if and only if $I^3 = QI^2$.*

Before closing this section, we briefly study the integrally closed ideal I with $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 3$. Suppose that $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 3$ then we have

$$0 < \ell_A(I^2/QI) \leq \mathrm{e}_1(I) - \mathrm{e}_0(I) + \ell_A(A/I) = 3$$

by Corollary 9. If $\ell_A(I^2/QI) = 1$ then we have $\mathrm{depth} G \geq d-1$ by [8, 13]. If $\ell_A(I^2/QI) = 3$ then the equality $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI)$ holds true, so that $I^3 = QI^2$ and the associated graded ring G of I is Cohen-Macaulay by Corollary 9. Thus we need to consider the following.

Theorem 17. *Suppose that $d \geq 2$. Assume that I is integrally closed and $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 3$ and $\ell_A(I^2/QI) = 2$. Let $c = \ell_A(I^3/QI^2)$. Then the following assertions hold true.*

(1) *Either $C \cong B(-2)$ as graded T-modules or there exists an exact sequence*

$$0 \to B(-3) \to B(-2) \oplus B(-2) \to C \to 0$$

of graded T-modules.

(2) *$1 \leq c \leq 2$ and $I^4 = QI^3$.*

(3) *Suppose $c = 1$ then* $\mathrm{depth} G \geq d - 1$ *and* $\mathrm{e}_2(I) = 4$, $\mathrm{e}_3(I) = 1$ *if $d \geq 3$, and* $\mathrm{e}_i(I) = 0$ *for $4 \leq i \leq d$.*

(4) *Suppose $c = 2$ then* $\mathrm{depth} G = d-2$ *and* $\mathrm{e}_2(I) = 3$, $\mathrm{e}_3(I) = -1$ *if $d \geq 3$,* $\mathrm{e}_4(I) = -1$ *if $d \geq 4$, and* $\mathrm{e}_i(I) = 0$ *for $5 \leq i \leq d$.*

(5) *The Hilbert series $HS_I(z)$ is given by*

$$HS_I(z) = \begin{cases} \dfrac{\ell_A(A/I) + \{\mathrm{e}_0(I) - \ell_A(A/I) - 2\}z + z^2 + z^3}{(1-z)^d}, & \text{if } c = 1, \\[2ex] \dfrac{\ell_A(A/I) + \{\mathrm{e}_0(I) - \ell_A(A/I) - 2\}z + 3z^3 - z^4}{(1-z)^d} & \text{if } c = 2. \end{cases}$$

We remark that $\ell_A(I^2/QI)$ measures how far is the multiplicity of I from the minimal value, see [9, Corollary 2.1]. If $\ell_A(I^2/QI) \leq 1$, then depth $G \geq d - 1$, but it is still open the problem whether $\mathrm{depth} G \geq d - 2$, assuming $\ell_A(I^2/QI) = 2$. Theorem 17 confirms the conjectured bound.

Corollary 18. *Assume that I is integrally closed. Suppose that* $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 3$. *Then* depth $G \geq d - 2$.

5. An Example

The goal of this section is to construct an example of Cohen-Macaulay local ring with the maximal ideal $\mathfrak{m}$ satisfying the equality in Theorem 2 (1). The class of examples we exhibit includes an interesting example given by H.-J. Wang, see [9, Example 3.2].

Theorem 19. *Let $d \geq c \geq 1$ be integers. Then there exists a Cohen-Macaulay local ring $(A, \mathfrak{m})$ such that*

$$d = \dim A, \quad \mathrm{e}_1(\mathfrak{m}) = \mathrm{e}_0(\mathfrak{m}) + \ell_A(\mathfrak{m}^2/Q\mathfrak{m}), \quad \text{and} \quad c = \ell_A(\mathfrak{m}^3/Q\mathfrak{m}^2)$$

for some minimal reduction $Q = (a_1, a_2, \cdots, a_d)$ of $\mathfrak{m}$.

To construct necessary examples we may assume that $c = d$.

Let $m \geq 0$ and $d \geq 1$ be integers. Let

$$D = k[[\{X_j\}_{1 \leq j \leq m}, Y, \{V_i\}_{1 \leq i \leq d}, \{Z_i\}_{1 \leq i \leq d}]]$$

be the formal power series ring with $m + 2d + 1$ indeterminates over an infinite field k, and let

$$\mathfrak{a} = [(X_j \mid 1 \leq j \leq m) + (Y)] \cdot [(X_j \mid 1 \leq j \leq m) + (Y) + (V_i \mid 1 \leq i \leq d)] \\ +(V_i V_j \mid 1 \leq i, j \leq d,\ i \neq j) + (V_i^3 - Z_i Y \mid 1 \leq i \leq d).$$

We set $A = D/\mathfrak{a}$ and denote the images of X_j, Y, V_i, and Z_i in A by x_j, y, v_i, and a_i, respectively. Then, since $\sqrt{\mathfrak{a}} = (X_j \mid 1 \leq j \leq m) + (Y) + (V_i \mid 1 \leq i \leq d)$, we have $\dim A = d$. Let $\mathfrak{m} = (x_j \mid 1 \leq j \leq m) + (y) + (v_i \mid 1 \leq i \leq d) + (a_i \mid 1 \leq i \leq d)$ be the maximal ideal in A and we set $Q = (a_i \mid 1 \leq i \leq d)$. Then, $\mathfrak{m}^2 = Q\mathfrak{m} + (v_i^2 \mid 1 \leq i \leq d)$, $\mathfrak{m}^3 = Q\mathfrak{m}^2 + Qy$, and $\mathfrak{m}^4 = Q\mathfrak{m}^3$. Therefore Q is a minimal reduction of $\mathfrak{m}$, and $a_1, a_2, \cdots, a_d$ is a system of parameters for A. We then have the following.

Theorem 20. *The following assertions hold true.*

(1) *A is a Cohen-Macaulay local ring with* $\dim A = d$.
(2) $\mathrm{C}_Q(\mathfrak{m}) \cong B_+(-1)$ *as graded T-modules. Therefore,* $\ell_A(\mathfrak{m}^3/Q\mathfrak{m}^2) = d$.
(3) $\mathrm{e}_0(\mathfrak{m}) = m + 2d + 2$, $\mathrm{e}_1(\mathfrak{m}) = m + 3d + 2$.
(4) $\mathrm{e}_2(\mathfrak{m}) = d + 1$ *if* $d \geq 2$, *and* $\mathrm{e}_i(\mathfrak{m}) = 0$ *for all* $3 \leq i \leq d$.
(5) $\mathrm{G}(\mathfrak{m})$ *is a Buchsbaum ring with* $\mathrm{depth}\mathrm{G}(\mathfrak{m}) = 0$ *and* $\mathbb{I}(\mathrm{G}(\mathfrak{m})) = d$.

References

[1] J. Elias and G. Valla, *Rigid Hilbert functions*, J. Pure and Appl. Algebra **71** (1991) 19–41.
[2] S. Goto, K. Nishida, and K. Ozeki, *Sally modules of rank one*, Michigan Math. J. **57** (2008) 359–381.
[3] A. Guerrieri and M. E. Rossi, *Hilbert coefficients of Hilbert filtrations*, J. Algebra **199** (1998) 40–61.
[4] C. Huneke, *Hilbert functions and symbolic powers*, Michigan Math. J. **34** (1987) 293–318.
[5] S. Itoh, *Hilbert coefficients of integrally closed ideals*, J. Algebra **176** (1995) 638–652.
[6] ———, *Integral closures of ideals generated by regular sequence*, J. Algebra **117** (1988) 390–401.
[7] D. G. Northcott, *A note on the coefficients of the abstract Hilbert function*, J. London Math. Soc. **35** (1960) 209–214.
[8] M. E. Rossi and G. Valla *A conjecture of J. Sally*, Comm. Alg. **24** (13) (1996) 4249–4261.
[9] ———.,*Hilbert functions of filtered modules*, UMI Lecture Notes 9, Springer (2010).
[10] J. D. Sally, *Ideals whose Hilbert function and Hilbert polynomial agree at $n = 1$*, J. Algebra **157** (1993), 534–547.
[11] W. V. Vasconcelos, *Hilbert Functions, Analytic Spread, and Koszul Homology*, Contemporary Mathematics, Vol **159** (1994) 410–422.
[12] M. Vaz Pinto, *Hilbert functions and Sally modules*, J. Algebra, **192** (1996) 504–523.
[13] H.-J. Wang, *Links of symbolic powers of prime ideals,* Math. Z. **256** (2007) 749–756.

Department of Mathematical Sciences
Faculty of Science
Yamaguchi University
1677-1 Yoshida Yamaguchi 753-8512 JAPAN
E-mail address: ozeki@yamaguchi-u.ac.jp