A GENERALIZATION OF DUAL SYMMETRY AND RECIPROCITY FOR SYMMETRIC ALGEBRAS

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Abstract. Slicing a module into semisimple ones is useful to study modules. Loewy structures provide a means of doing so. To establish the Loewy structures of projective modules over a finite dimensional symmetric algebra over a field F, the Landrock lemma is a primary tool. The lemma and its corollary relate radical layers of projective indecomposable modules to radical layers of the F-duals of those modules ("dual symmetry") and to socle layers of those modules ("reciprocity").

We generalize these results to artin algebras in functorial manner. Our main theorem relates radical layers of projective modules to socle layers of injective modules. A key tool to prove the main theorem is a pair of adjoint functors, which we call socle functors and capital functors.

Key Words: Loewy structure, radical layer, socle layer, symmetric algebra, Landrock's lemma.

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1. Introduction

Semisimple modules are one of the most well-understood classes of modules. Hence slicing a module into semisimple ones is a natural way to study modules. Loewy structures provide a means of doing so. To establish Loewy structures several studies has been done [2, 3, 8]. A primary tool in these studies is the Landrock lemma [4], which is stated below.

Let A be an R-artin algebra and

$$(-)^* \coloneqq \operatorname{Hom}_R(-, E)$$

the *standard duality* where E is the minimal injective cogenerator of $\operatorname{mod} R$, the category of finitely generated (right) R-modules. The opposite algebra is denoted by A^{op}. The term module refers to a finitely generated right module. Recall that A is a *symmetric algebra* if $A \cong A^*$ as (A, A)-bimodules. For other notations see Definition 6.

Theorem 1 (Landrock [4, Theorem B]). *For a finite dimensional symmetric algebra A over a field F, let P_i and P_j be the projective covers of simple A-modules S_i and S_j respectively. Then for every integer $n \geq 1$ we have an F-linear isomorphism*

$$\operatorname{Hom}_A(\operatorname{rad}_n P_i, S_j) \cong \operatorname{Hom}_{A^{\mathrm{op}}}\left(\operatorname{rad}_n(P_j^*), S_i^*\right).$$

The detailed version of this paper will be submitted for publication elsewhere.

Corollary 2. *Under the same assumptions and notations of Theorem* 1*, we have an F-linear isomorphism*

$$\operatorname{Hom}_A(\operatorname{rad}_n P_i, S_j) \cong \operatorname{Hom}_A(S_i, \operatorname{soc}_n P_j).$$

Although these results are powerful as indicated at the beginning, they are not applicable to algebras other than finite dimensional symmetric ones. We generalize the above results to artin algebras. To state our main theorem we let

$$(-)^\vee := \operatorname{Hom}_A(-, A)$$

be the A-*dual functor* and $\nu(-) := \big((-)^\vee\big)^*$ the *Nakayama functor*. Let $\operatorname{proj} A$ denotes the category of projective A-modules and $\operatorname{inj} A$ the category of injective A-modules.

Theorem 3. *Let A be an artin R-algebra and n a natural number. Then two functors*

$$\operatorname{Hom}_A(\operatorname{cap}_n -, \nu ?), \operatorname{Hom}_A(-, \operatorname{soc}_n \nu ?) \colon (\operatorname{proj} A)^{\mathrm{op}} \times \operatorname{proj} A \to \operatorname{mod} R$$

are naturally isomorphic.

Corollary 4 (Dual of Theorem 3)**.** *Let A be an artin R-algebra and n a natural number. Then two functors*

$$\operatorname{Hom}_A(\nu^{-1} -, \operatorname{soc}_n ?), \operatorname{Hom}_A(\operatorname{cap}_n \nu^{-1} -, ?) \colon (\operatorname{inj} A)^{\mathrm{op}} \times \operatorname{inj} A \to \operatorname{mod} R$$

are naturally isomorphic.

Remark 5*.* The above results are slightly generalized than [7] reflecting the comments and questions the author got in this symposium.

2. Notation

We introduce basic terminology of this proceeding and state some useful lemmas first. Then proofs of the main theorem is given. The Landrock lemma is proved as a special case of the main theorem.

Definition 6. For a module V over an algebra, $\operatorname{soc} V$ denotes the sum of minimal submodules of V and $\operatorname{rad} V$ denotes the intersection of maximal submodules of V. For an integer $n \geq 0$, the *nth socle* of V is defined inductively by $\operatorname{soc}^0 V = 0$ and

$$\operatorname{soc}^n V = \{\, v \in V \mid v + \operatorname{soc}^{n-1} V \in \operatorname{soc}(V/\operatorname{soc}^{n-1} V) \,\} \tag{2.1}$$

if $n > 0$. For an integer $n \geq 0$, the *nth radical* of V is also defined inductively by $\operatorname{rad}^0 V = V$ and

$$\operatorname{rad}^n V = \operatorname{rad}(\operatorname{rad}^{n-1} V) \tag{2.2}$$

if $n > 0$. We then write

$$\operatorname{soc}_n V = \operatorname{soc}^n V / \operatorname{soc}^{n-1} V \tag{2.3}$$

and call it the *nth socle layer* of V for $n \geq 1$. We also write

$$\operatorname{rad}_n V = \operatorname{rad}^{n-1} V / \operatorname{rad}^n V \tag{2.4}$$

and call it the *nth radical layer* of V for $n \geq 1$.

Definition 7. For an integer $n \geq 0$ and a module V over an algebra we write $\operatorname{cap}^n V = V/\operatorname{rad}^n V$ and call it the *nth capital* of V. Since every homomorphism maps the nth socle into the nth socle and the nth radical into the nth radical, soc^n and cap^n define endofunctors. We call these endofunctors the *nth socle functor* and the *nth capital functor* respectively.

3. Sketch of Proofs

We refer the reader to [7] for details. (Although proofs are given for finite dimensional cases almost the same argument work.) The next simple lemma, which is essentially the hom-tensor adjunction, is vital to prove the main theorem.

Lemma 8 ([7, Lemma 2.3])**.** *Let A be an artin algebra. Then for every integer $n \geq 0$ the nth capital functor and nth socle functor yield an adjoint pair of functors.*

$$\operatorname{mod} A \underset{\operatorname{soc}^n}{\overset{\operatorname{cap}^n}{\rightleftarrows}} \operatorname{mod} A \qquad \operatorname{cap}^n \dashv \operatorname{soc}^n .$$

For a sense of unity we adopt an alias

$$\operatorname{cap}_n = \operatorname{rad}_n . \tag{3.1}$$

Note that cap_n and soc_n define endofunctors as cap^n and soc^n. The following lemma can essentially be found in [5, Problem 2.14.ii].

Lemma 9 ([7, Lemma 2.4])**.** *Let A be an artin algebra. For every integer $n \geq 1$ we have a natural isomorphism*

$$\begin{tikzcd} \operatorname{mod} A \arrow[loop left, "\operatorname{soc}_n"] \arrow[r, "(-)^*"] & \operatorname{mod} A^{\mathrm{op}} \arrow[loop right, "\operatorname{cap}_n"] \end{tikzcd} \qquad (\operatorname{soc}_n(-))^* \cong \operatorname{cap}_n((-)^*).$$

Sketch of Proof of Theorem 3. Let $(P, Q), (P', Q') \in (\operatorname{proj} A)^{\mathrm{op}} \times \operatorname{proj} A$ and take a morphism $(P, Q) \to (P', Q')$. Then we have the following four short exact sequences.

$$\begin{tikzcd}[column sep=tiny]
\operatorname{Hom}(\operatorname{cap}^{n-1} P, \nu Q) \arrow[rr, hook] \arrow[dr, dashed] \arrow[dd, dashed] & & \operatorname{Hom}(\operatorname{cap}^{n} P, \nu Q) \arrow[rr, two heads] \arrow[dr, dashed] \arrow[dd, dashed] & & \operatorname{Hom}(\operatorname{cap}_{n} P, \nu Q) \arrow[dr, dashed] \arrow[dd, dashed] & \\
& \operatorname{Hom}(P, \operatorname{soc}^{n-1} \nu Q) \arrow[rr, hook] \arrow[dd, dashed] & & \operatorname{Hom}(P, \operatorname{soc}^{n} \nu Q) \arrow[rr, two heads] \arrow[dd, dashed] & & \operatorname{Hom}(P, \operatorname{soc}_{n} \nu Q) \arrow[dd, dashed] \\
\operatorname{Hom}(\operatorname{cap}^{n-1} P', \nu Q') \arrow[rr, hook] \arrow[dr, dashed] & & \operatorname{Hom}(\operatorname{cap}^{n} P', \nu Q') \arrow[rr, two heads] \arrow[dr, dashed] & & \operatorname{Hom}(\operatorname{cap}_{n} P', \nu Q') \arrow[dr, dashed] & \\
& \operatorname{Hom}(P', \operatorname{soc}^{n-1} \nu Q') \arrow[rr, hook] & & \operatorname{Hom}(P', \operatorname{soc}^{n} \nu Q') \arrow[rr, two heads] & & \operatorname{Hom}(P', \operatorname{soc}_{n} \nu Q')
\end{tikzcd}$$

Then we have the left four diagonal dashed morphisms by Lemma 8 and have the right two diagonal dashed morphisms that commute top and bottom faces of the right cuboid. In addition, the two morphisms are isomorphisms by the five lemma. We also have vertical dashed morphisms that canonically defined. Finally a chase revels that the right face of the right cuboid commute. Thus we have

$$\operatorname{Hom}_A(\operatorname{cap}_n P, \nu Q) \cong \operatorname{Hom}_A(P, \operatorname{soc}_n \nu Q). \tag{3.2}$$ □

Remark 10. The above proof is element-free.

We derive Theorem 1 and Corollary 2 from the main theorem in the following.

Corollary 11 ([7, Theorem 1.3])**.** *For an artin R-algebra A, let P_i and P_j be the projective covers of simple A-modules S_i and S_j respectively. Then for every integer $n \geq 1$ we have R-linear isomorphisms*

$$\operatorname{Hom}_A(\operatorname{rad}_n P_i, S_j) \cong \operatorname{Hom}_{A^{\mathrm{op}}}\left(\operatorname{rad}_n(P_j^\vee), S_i^*\right) \tag{3.3}$$

and

$$\operatorname{Hom}_A(\operatorname{rad}_n P_i, S_j) \cong \operatorname{Hom}_A(S_i, \operatorname{soc}_n \nu P_j). \tag{3.4}$$

The next properties of symmetric algebras is well-known.

Proposition 12 ([1, Proposition IV.3.8])**.** *For every symmetric artin algebra, we have the following.*

(i) $(-)^* \cong (-)^\vee$.
(ii) $\nu \cong 1$.

Proof of Theorem 1. Apply Proposition 12(i) to (3.3). □

Proof of Corollary 2. Apply Proposition 12(ii) to (3.4). □

Remark 13. Okuyama and Tsushima gave a short proof of the Landrock lemma for group algebras in [6, Theorem 2].

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