REPRESENTATIONS OF SMALL CATEGORIES AND GROTHENDIECK DERIVATORS

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Abstract. We explain the notion of stable derivator and how it enhances triangulated categories. We also show how classical derived equivalences based on Bernstein-Gelfand-Ponomarev reflection functors generalize to a refined version in the context of stable derivators, and indicate what tilting modules are in that context.

1. Introduction

The concept of derivator appeared independently in the work of Heller [13], Grothendieck [4] and Franke [3]. One motivation for using derivators is that they provide an enhancement of triangulated categories which fixes some well-known problems, such as non-functoriality of the mapping cone construction. The main favorable features are that the language is still relatively elementary (it employs standard category theory) and that the setting is homotopy invariant (which in algebra typically simply means that quasi-isomorphisms are always made formally invertible).

In order to explain the philosophy, let us inspect the issue with the non-functoriality of mapping cones. To this end, we will denote by $[n]$ the linearly ordered set $(0 < 1 < \cdots < n)$. Now consider an abelian category $\mathcal{A}$ and the category $\mathcal{A}^{[1]}$ of morphism in $\mathcal{A}$. We have the right exact cokernel functor $\mathsf{cok}\colon \mathcal{A}^{[1]} \to \mathcal{A}$ and its left derived functor is precisely the mapping cone. To be more precise, the cokernel lifts to categories of complexes, $\mathsf{cok}\colon \mathsf{Ch}(\mathcal{A}^{[1]}) \cong \mathsf{Ch}(\mathcal{A})^{[1]} \to \mathsf{Ch}(\mathcal{A})$ and we have a square with a natural transformation

$$\begin{tikzcd} \mathsf{Ch}(\mathcal{A}^{[1]}) \arrow[r, "\mathsf{cok}"] \arrow[d] & \mathsf{Ch}(\mathcal{A}) \arrow[d] \\ \mathsf{D}(\mathcal{A}^{[1]}) \arrow[r, "\mathsf{cone}"'] \arrow[ur, phantom, "\Nearrow"] & \mathsf{D}(\mathcal{A}) \end{tikzcd}$$

satisfying a universal property (we refer to [14, §8.4–8.5] for details). As usual, this is proved by taking suitable left resolutions. It suffices to observe that

(1) the cokernel functor $\mathsf{cok}\colon \mathsf{Ch}(\mathcal{A}^{[1]}) \to \mathsf{Ch}(\mathcal{A})$ is exact on the full exact subcategory of $\mathsf{Ch}(\mathcal{A}^{[1]})$ consisting of monomorphisms (by the 3×3-lemma),
(2) each morphism $f\colon X \to Y$ admits a surjective quasi-isomorphism from the monomorphism $X \to Y \oplus CX$ (here CX is just the cone of $1_X\colon X \to X$), and
(3) the cokernel of $X \to Y \oplus CX$ is none other than the $\mathsf{cone}(f)$.

This is a short introduction to motivation and results of the series of joint papers [8, 9, 10, 11] with Moritz Groth.

Here is the subtlety: The mapping cone construction yields a functor $\mathsf{D}(\mathcal{A}^{[1]}) \to \mathsf{D}(\mathcal{A})$ but this functor does *not* factor through the canonical functor $\mathsf{D}(\mathcal{A}^{[1]}) \to \mathsf{D}(\mathcal{A})^{[1]}$ which we denote by $\mathsf{dia}_{[1]}$. The derived category of the morphism category is simply not equivalent to the morphism category of the derived category—$\mathsf{dia}_{[1]}$ is always full and essentially surjective, but rarely faithful. One can see that explicitly already for $\mathcal{A} = \mathsf{Mod}\, k$ where k is a field. The category of compact objects of $\mathsf{D}(\mathcal{A}^{[1]}) \simeq \mathsf{D}(\mathsf{Mod}\, k(\bullet \to \bullet))$ has the Auslander-Reiten quiver

$$\begin{tikzcd}[column sep=small]
 & \Sigma^{-1}(k \to 0) \arrow[dr, "\Sigma^{-1}\delta"] & & (k \to k) \arrow[dr] & & \Sigma(0 \to k) \arrow[dr] & \\
\dots \arrow[ur] & & (0 \to k) \arrow[ur] & & (k \to 0) \arrow[ur, "\delta"] & & \dots
\end{tikzcd}$$

A direct computation reveals that cone is a localization functor which sends all suspensions of δ to isomorphisms, whereas $\mathsf{dia}_{[1]}$ is an additive quotient functor which sends all suspensions of δ to zero.

The conclusion is that if we wish to have a functorial cone, we better consider not only $\mathsf{D}(\mathcal{A})$ alone, but also $\mathsf{D}(\mathcal{A}^{[1]})$, the derived category of morphisms. If we want other functorial derived limit and colimit constructions (e.g. homotopy pushouts or pullbacks), we should consider representations in $\mathcal{A}$ of other shapes too. In fact, it is often easiest to consider *all* shapes, i.e. the assignment

$$\mathsf{Cat} \ni C \quad \longmapsto \quad \mathsf{D}(\mathcal{A}^C),$$

together with various functors and natural transformations connecting these derived categories. This is very roughly what a derivator is and this is also why representation theory is relevant—we consider representations of categories C in $\mathcal{A}$.

2. Kan extensions

As indicated, another important ingredient of the definition of a derivator are derived limits and colimits. Given $C \in \mathsf{Cat}$, there is an exact 'constant diagram' functor $\Delta_C \colon \mathcal{A} \to \mathcal{A}^C$. The usual functors of limits and colimits of C-shaped diagrams, $\mathcal{A}^C \to \mathcal{A}$, if they exist, are then right and left adjoints to Δ_C, respectively.

Homotopy limits and *homotopy colimits* are simply derived versions, provided that they exist. As this usually does not cause any confusion, we will omit the word 'homotopy'.

$$\begin{tikzcd}
\mathsf{D}(\mathcal{A}^C) \arrow[r, bend left, "\mathsf{colim}"] \arrow[r, bend right, "\mathsf{lim}"'] & \mathsf{D}(\mathcal{A}) \arrow[l, "\Delta_C"']
\end{tikzcd}$$

There is a technical point, though. We often need more than just colimits, namely so-called Kan extensions. If $X \colon C \to \mathcal{A}$ is an object of $\mathcal{A}^C$ and $u \colon C \to D$ is a functor in Cat, we might want to extend X to an object of $\mathcal{A}^D$. We usually cannot extend X on the nose, but only up to a natural transformation satisfying a universal property. As there is a choice regarding the direction of the transformation, we distinguish between left and right Kan extensions.

$$\begin{tikzcd} C \arrow[r, "X"] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow\eta"] & \mathcal{A} \\ D \arrow[ur, dotted, "\mathrm{LKan}_u(X)"'] & \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "X"] \arrow[d, "u"'] \arrow[dr, phantom, "\varepsilon\Uparrow"] & \mathcal{A} \\ D \arrow[ur, dotted, "\mathrm{RKan}_u(X)"'] & \end{tikzcd}$$

Standard facts about left Kan extensions are summarized in the following proposition. A dual version holds for right Kan extensions. We denote by $\mathbb{1}$ the terminal category with one object and its identity morphism only, and if $Y \in \mathcal{A}^D$ and $d \in D$, we denote by $Y_d \in \mathcal{A}$ the component $Y(d)$ of Y at the object d.

Proposition 1. *Let $\mathcal{A}$ be a cocomplete abelian category and let $u\colon C \to D$ be a functor. Then the following hold:*

(1) $(\mathrm{LKan}_u(X), \eta)$ *exists for each $X \in \mathcal{A}^C$;*
(2) $\mathrm{LKan}_u\colon \mathcal{A}^C \to \mathcal{A}^D$ *is a left adjoint functor to the forgetful functor $u^*\colon \mathcal{A}^D \to \mathcal{A}^C$ given by $X \mapsto X \circ u$ (hence C-shaped colimits are simply left Kan extensions along $C \to \mathbb{1}$);*
(3) *given $X \in \mathcal{A}^C$ and $d \in D$, we have*

$$\mathrm{LKan}_u(X)_d \cong \mathsf{colim}_{u/d}(X \circ p)$$

via a canonical morphism, where u/d is a slice category whose objects are of the form $(c \in C, g\colon u(c) \to d)$, and $p\colon u/d \to C$ is the canonical projection;
(4) *if $u\colon C \to D$ is fully faithful, so is $\mathrm{LKan}_u\colon \mathcal{A}^C \to \mathcal{A}^D$.*

Example 2. To illustrate the distinction between Kan extension and colimits, consider the commutative square $\square = [1] \times [1]$, the embedding of the upper left corner $i_{\ulcorner}\colon \ulcorner \to \square$, and the unique functor $\pi\colon \ulcorner \to \mathbb{1}$. If $X \in \mathcal{A}^{\ulcorner}$, then $\mathrm{LKan}_\pi \in \mathcal{A}(X)$ is the pushout of X as an object, while $\mathrm{LKan}_{i_{\ulcorner}}(X) \in \mathcal{A}^{\square}$ is the corresponding pushout square.

As with limits and colimits, we also consider the derived versions if they exist. Following [4, 5], we will use the shorthand notations $u_!$ for the (derived) left Kan extensions and u_* for the (derived) right Kan extension.

3. Derivators (definition and examples)

Now we can formally define our main object of interest.

Definition 3. A *prederivator* $\mathscr{D}$ is a strict 2-functor $\mathscr{D}\colon \mathsf{Cat}^{\mathrm{op}} \to \mathsf{CAT}$, where Cat is the 2-category of all small categories and CAT is the '2-category' of all (not necessarily small) categories.

Remark 4. We follow the convention of [5] that 'op' in $\mathsf{Cat}^{\mathrm{op}}$ applies only to functors and not to natural transformations. That is

$$\mathscr{D}\colon \quad \begin{tikzcd} C \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow\alpha"] & D \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} \mathscr{D}(C) & \mathscr{D}(D) \arrow[l, bend right, "v^*"'] \arrow[l, bend left, "u^*"] \arrow[l, phantom, "\Downarrow\alpha^*"] \end{tikzcd}$$

There is no agreement on that in the literature, [4] changes the direction of natural transformations as well for example.

Note also that we use the notation u^* for $\mathscr{D}(u)$ and α^* for $\mathscr{D}(\alpha)$.

A derivator is a prederivator which satisfies basic properties of derived Kan extensions, analogous to Proposition 1, parts (2) and (3).

Definition 5. A derivator $\mathscr{D}$ is a prederivator satisfying the following axioms:

(Der1) For each collection $(C_i)_{i\in I}$ of small categories, the canonical functor $\mathscr{D}(\coprod_{i\in I} C_i) \to \prod_{i\in I}\mathscr{D}(C_i)$ is an equivalence.

(Der2) Given $C \in \mathsf{Cat}$ and $f\colon X \to Y$ in $\mathscr{D}(C)$, then f is an isomorphism if and only if $f_c\colon X_c \to Y_c$ is an isomorphism in $\mathscr{D}(\mathbb{1})$ for every $c \in C$. Here $X_c \in \mathscr{D}(\mathbb{1})$ is just a shorthand notation for $c^*(X)$, where we identify c with the functor $\mathbb{1} \to C$ which points at the object c.

(Der3) For each $u\colon C \to D$ in Cat, the functor $u^*\colon \mathscr{D}(D) \to \mathscr{D}(C)$ has both adjoints.

$$\begin{tikzcd} \mathscr{D}(C) \arrow[r, bend left, "u_! = \mathrm{LKan}_u"] \arrow[r, bend right, "u_* = \mathrm{RKan}_u"'] & \mathscr{D}(D) \arrow[l, "u^*"'] \end{tikzcd}$$

(Der4) Pointwise formulas for $u_!(X)_d$ and $u_*(X)_d$ analogous to Proposition 1(3) and its dual hold (see [5, §1.1] for details).

Objects $X \in \mathscr{D}(C)$ are called *coherent C-shaped diagrams* in $\mathscr{D}$ (as opposed to *incoherent diagrams*, which are simply C-shaped diagrams in $\mathscr{D}(\mathbb{1})$). The 2-functor structure of $\mathscr{D}$ allows to compare between these two notions. Namely, one can construct an abstract *underlying diagram functor*

$$\mathsf{dia}_C\colon \mathscr{D}(C) \to \mathscr{D}(\mathbb{1})^C.$$

Example 6. (1) If $\mathcal{A}$ is a category, then $\mathscr{Y}_\mathcal{A}\colon C \longmapsto \mathcal{A}^C$ is a prederivator, and it is a derivator if and only if $\mathcal{A}$ is complete.

(2) If $\mathcal{A}$ is a Grothendieck abelian category, $\mathscr{D}_\mathcal{A}\colon C \longmapsto \mathsf{D}(\mathcal{A}^C)$ is a derivator, which enhances the usual derived category $\mathsf{D}(\mathcal{A}) = \mathscr{D}_\mathcal{A}(\mathbb{1})$.

(3) More generally, given any model category $\mathcal{M}$, there is a homotopy derivator $\mathscr{H}\!o_\mathcal{M}\colon C \longmapsto \mathsf{Ho}(\mathcal{M}^C)$ (see [1] and [5, Proposition 1.30]).

Remark 7. One can generalize various facts from ordinary category theory to the derivator setting. For example, if $u\colon C \to D$ is fully faithful, then $u_!, u_*\colon \mathscr{D}(C) \to \mathscr{D}(D)$ are fully faithful functors for any derivator $\mathscr{D}$ (see [5, Proposition 1.20] and compare to Proposition 1(4)).

The latter observation allows for the following definition (cp. Example 2).

Definition 8. Consider the fully faithful embeddings $i_\ulcorner\colon \ulcorner \to \square$ and $i_\lrcorner\colon \lrcorner \to \square$ of corners into a commutative square $\square = [1]\times[1]$. Objects in the essential image of the fully faithful functor $(i_\ulcorner)_!\colon \mathscr{D}(\ulcorner) \to \mathscr{D}(\square)$ are called *coherent cocartesian squares*. Dually, objects in the essential image of $(i_\lrcorner)_*$ are *coherent cartesian squares*.

4. Pointed and stable derivators

As the examples above show, a general derivator does not have much to do with triangulated categories. For that we need additional axioms.

Definition 9. A derivator $\mathscr{D}$ is pointed if the underlying category $\mathscr{D}(\mathbb{1})$ has a zero (= simultaneously initial and terminal) object $0 \in \mathscr{D}(\mathbb{1})$.

One can show that then $\mathscr{D}(C)$ has a zero object for each $C \in \mathsf{Cat}$, [5, Proposition 3.2]. Derivators $\mathscr{D}_{\mathcal{A}}$ of Grothendieck abelian categories and, more generally, homotopy derivators $\mathscr{H}\!o_{\mathcal{M}}$ of pointed model categories serve as examples.

With pointed derivators one can mimic standard constructions of fiber and cofiber sequences from algebraic topology. In the context of $\mathscr{D}_{\mathcal{A}}$ for $\mathcal{A}$ Grothendieck abelian, one precisely constructs exact triangles this way. To see how this works, consider the fully faithful functors in Cat, where the image of i is the upper left horizontal morphism of the category in the middle:

$$
\begin{tikzcd} \bullet \arrow[r] & \bullet \end{tikzcd}
\quad \xrightarrow{i} \quad
\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] & \bullet \\ \bullet & & \end{tikzcd}
\quad \xrightarrow{j} \quad
\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "="] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "="] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}
$$

We consider the action of the fully faithful functor $j_! \circ i_* \colon \mathscr{D}([1]) \to \mathscr{D}([2] \times [1])$. If $F \in \mathscr{D}([1])$ is a coherent morphism whose underlying incoherent morphism is $\mathsf{dia}_{[1]}(F) = f \colon x \to y$, the underlying diagram of $j_!(i_*(F))$ in $\mathscr{D}(\mathbb{1})$ looks like the left hand side of

$$
\begin{tikzcd} x \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "="] & y \arrow[r] \arrow[d, "g"] \arrow[dr, phantom, "="] & 0 \arrow[d] \\ 0 \arrow[r] & \mathsf{cone}(f) \arrow[r, "h"] & x' \end{tikzcd}
\qquad\qquad
\begin{tikzcd} x \arrow[r] \arrow[d] \arrow[dr, phantom, "="] & 0 \arrow[d] \\ 0 \arrow[r] & x' \end{tikzcd}
\tag{4.1}
$$

One can prove from the axioms that the two small squares in $j_!(i_*(F))$ are cocartesian, and so is the outer one, depicted on the right hand side of (4.1). In the ordinary category theory (or in a derivator of the form $\mathscr{Y}_{\mathcal{A}}$ as in Example 6(1)), this would imply $x' = 0$. However, already in the case of Example 6(2), we have $x' \cong \Sigma x$. In fact, in any pointed derivator $\mathscr{D}$ we can define the suspension functor

$$\Sigma \colon \mathscr{D}(\mathbb{1}) \longrightarrow \mathscr{D}(\mathbb{1})$$

using the right hand side square of (4.1). As the suspension is defined using adjoint functors, it is determined within $\mathscr{D}$ by a universal property, and so it is unique up to a unique isomorphism. By identifying x' with Σx in (4.1) and taking the corresponding subdiagram, we obtain an incoherent cofiber sequence in $\mathscr{D}(\mathbb{1})$:

$$x \xrightarrow{f} y \xrightarrow{g} \mathsf{cone}(f) \xrightarrow{h} \Sigma x\,. \tag{4.2}$$

However, this still does not mean that we have a triangulated category. For example, consider the standard model structure on Top_*, the category of pointed topological spaces, where the class of morphisms to be formally made invertible are the usual weak homotopy equivalences. Then the derivator $\mathscr{H}\!o_{\mathsf{Top}_*}$ has the usual homotopy category of pointed spaces as the underlying category, and this category is not even additive. To fix that, we need so-called stable derivators.

Definition 10. A pointed derivator is *stable*, if it satisfies either of the following two equivalent conditions (see [6, Theorem 7.1]):

(1) The suspension functor Σ is an equivalence.
(2) A coherent square $X \in \mathscr{D}(\square)$ is cartesian if and only if it is cocartesian.

Now we have the following result (it was sketched by Maltsiniotis in [16], with details filled in by Groth in [5, §4]).

Theorem 11. *Let $\mathscr{D}$ be a strong stable derivator. Then $\mathscr{D}(\mathbb{1})$ with the suspension functor and cofiber sequences as described above forms a triangulated category.*

Remark 12. To construct a cofiber sequence (4.2), one needs to start with a coherent morphism $F \in \mathscr{D}([1])$. Since the triangulated structure requires a triangle for any (incoherent) morphism $f \in \mathscr{D}(\mathbb{1})^{[1]}$, we need to be able to lift incoherent morphisms to coherent ones. This is roughly what a *strong* derivator is. More precisely, we require for technical reasons that all (naturally defined) partial diagram functors $\mathsf{dia}_{A,[1]} \colon \mathscr{D}(A \times [1]) \to \mathscr{D}(A)^{[1]}$ are full and essentially surjective (these functors also reflect isomorphisms by (Der2)).

Remark 13. In fact, more was proved in [5, §4] under the assumptions of Theorem 11—$\mathscr{D}(C)$ is triangulated for each C and $u^*, u_!, u_*$ are exact functors for each $u \colon C \to D$. Even better, $\mathscr{D}$ lifts to a 2-functor whose codomain is the 'category' of triangulated categories.

$$\begin{tikzcd} & \mathsf{TriaCAT} \arrow[d, "\text{forget}"] \\ \mathsf{Cat}^{\mathrm{op}} \arrow[r, "\mathscr{D}"'] \arrow[ur, dotted] & \mathsf{CAT} \end{tikzcd}$$

5. Abstract reflection functors

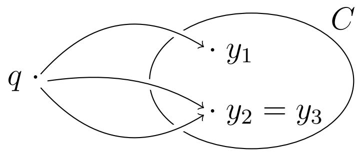
If k is a field, the strong stable derivator $\mathscr{D}_k := \mathscr{D}_{\mathsf{Mod}\, k}$ was in fact studied a lot in representation theory. This is of course since if C is a category with finitely many objects, then $\mathscr{D}_k(C) = \mathsf{D}((\mathsf{Mod}\, k)^C) \simeq \mathsf{D}(\mathsf{Mod}\, kC)$, where kC is the category algebra over k.

A well-known phenomenon going back to Happel [12] is that there exist a plethora of nontrivial derived equivalences $\mathscr{D}_k(C) \simeq \mathscr{D}_k(D)$ for non-equivalent pairs of categories $C \not\simeq D$. The simplest of them is wired in the definition of a stable derivator. Since cocartesian squares are cartesian and vice versa, the composition of the following adjunctions yields a pair of mutually inverse triangle equivalences

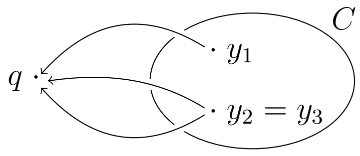
$$\begin{tikzcd} \mathscr{D}_k(\bullet \leftarrow \bullet \rightarrow \bullet) \arrow[r, shift left, "(i_{\ulcorner})_!"] & \mathscr{D}_k(\square) \arrow[l, shift left, "i_{\ulcorner}^*"] \arrow[r, shift left, "i_{\lrcorner}^*"] & \mathscr{D}_k(\bullet \rightarrow \bullet \leftarrow \bullet) \arrow[l, shift left, "(i_{\lrcorner})_*"] \end{tikzcd} .$$

Of course such an equivalence exist for *every* stable derivator by the very definition. By pushing this idea further, many other derived equivalences based on reflection functors generalize to an arbitrary stable derivator as well. Working out this was one of the aims of the series of papers [8, 9, 10, 11]. There is also a previous work by Ladkani [15], which is conceptually closely related, but using rather different methods.

In [11], we start with an arbitrary small category $C \in \mathsf{Cat}$ and finitely many (not necessarily distinct) objects $y_1, \ldots, y_n \in C$. Then we form categories C^- and C^+ by adding freely to C a new object q and n new morphisms pointing to or from, respectively, each of $y_1, \ldots, y_n$.



The category C^-



The category C^+

Then $\mathscr{D}(C^-) \simeq \mathscr{D}(C^+)$ for any stable derivator by [11], but in fact more was proved there. Given any derivator $\mathscr{D}$ and $A \in \mathsf{Cat}$, there exists a *shifted derivator* given by

$$\mathscr{D}^A\colon C \longmapsto \mathscr{D}(A \times C).$$

Furthermore, derivators themselves form a '2-category' DER, [5, §2]. Of particular interest is the notion of an *exact morphism* $F\colon \mathscr{D} \to \mathscr{E}$ of stable derivators. By definition, F must preserve zero objects and cartesian/cocartesian squares. If $\mathscr{D}, \mathscr{E}$ are strong and stable, exact morphisms $F\colon \mathscr{D} \to \mathscr{E}$ in DER can be viewed as enhancements of exact functors of the underlying triangulated categories. The main result is:

Theorem 14 ([11, Theorem 9.11])**.** *Let $C \in \mathsf{Cat}$ and $y_1, \ldots, y_n \in C$. Then for any stable derivator $\mathscr{D}$, there exists an equivalence*

$$\Phi_{\mathscr{D}}\colon \mathscr{D}^{C^-} \xrightarrow{\sim} \mathscr{D}^{C^+}.$$

Moreover, given any exact morphism $F\colon \mathscr{D} \to \mathscr{E}$ of stable derivators, there is a natural isomorphism $\gamma_F\colon F^{C^+} \circ \Phi_{\mathscr{D}} \to \Phi_{\mathscr{E}} \circ F^{C^-}$. These natural isomorphisms satisfy certain coherence relations making them compatible with compositions of exact morphisms of stable derivators and with natural transformations between exact morphisms of stable derivators.

This improves classical derived equivalences based on reflection functors in several ways. We obtain a more general result (for any stable derivator), more refined equivalences (derivator equivalences as opposed to triangulated equivalences) and more compatibility (the equivalences commute with any other exact morphism of stable derivators).

6. Tilting bimodules

Following [10, §8], we can encode for a given pair C^+, C^- the entire package $(F_{\mathscr{D}}, \gamma_F)$ of derivator equivalences and natural isomorphisms from Theorem 14 into a single tilting bimodule. This bimodule is quite explicit and resembles classical tilting bimodules representing Happel's derived equivalences, only the coefficients are not in a field k but rather in the sphere spectrum.

The key point is related to a derivator $\mathscr{H}o_{\mathsf{Sp}}$ enhancing the stable homotopy category of spectra. The smash product of spectra induces a monoidal structure on $\mathscr{H}o_{\mathsf{Sp}}$ in the sense of [7]. That is, we for any pair $A, B \in \mathsf{Cat}$ we have a functor

$$\otimes\colon \mathscr{H}o_{\mathsf{Sp}}(A) \times \mathscr{H}o_{\mathsf{Sp}}(B) \to \mathscr{H}o_{\mathsf{Sp}}(A \times B)$$

compatible with the 2-functor structure of $\mathscr{H}o_{\mathsf{Sp}}$ and commuting with (homotopy) colimits in each variable. The key result for us is that by [2, §A.3], every stable derivator $\mathscr{D}$ admits a canonical action by $\mathscr{H}o_{\mathsf{Sp}}$, i.e. we have

$$\otimes\colon \mathscr{H}o_{\mathsf{Sp}}(A) \times \mathscr{D}(B) \to \mathscr{D}(A \times B)$$

with similar properties. Moreover, for any category C, there is a relatively straightforward way to define the 'tensor product over C', i.e. functors

$$\otimes_{[C]}\colon \mathscr{H}o_{\mathsf{Sp}}(A \times C^{\mathrm{op}}) \times \mathscr{D}(C \times B) \to \mathscr{D}(A \times B)$$

(see [7, §5]). Now, in the situation of Theorem 14, the techniques of [10, §8] allow us to find an object $T \in \mathscr{H}o_{\mathsf{Sp}}(C^+ \times (C^-)^{\mathrm{op}})$ (this is our tilting bimodule) such that for every stable derivator $\mathscr{D}$ we have

$$F_{\mathscr{D}} \cong T \otimes_{[C^-]} -\colon \mathscr{D}^{C^-} \longrightarrow \mathscr{D}^{C^+}.$$

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