3-DIMENSIONAL CUBIC CALABI-YAU ALGEBRAS AND SUPERPOTENTIALS

KENTA UEYAMA

Abstract. Recently, Mori and Smith studied 3-dimensional noetherian quadratic Calabi-Yau algebras using superpotentials. As a continuation, in this note, we study 3-dimensional noetherian cubic Calabi-Yau algebras using superpotentials. The main result of this note is to classify all superpotentials whose Jacobian algebras are 3-dimensional noetherian cubic Calabi-Yau algebras. As an application, we compute the homological determinants of graded algebra automorphisms of 3-dimensional noetherian cubic Calabi-Yau algebras.

1. Calabi-Yau Algebras and Superpotentials

This note is based on a joint work [10] with Izuru Mori.

Throughout this note, let k be an algebraically closed field of characteristic 0, and V a finite dimensional vector space over k. A graded algebra means a graded algebra finitely generated in degree 1 over k, that is, every graded algebra is of the form $A = T(V)/I$ where $T(V)$ is the tensor algebra on V over k, and I is a two-sided ideal of $T(V)$. We fix a basis $\{x_1, \dots, x_n\}$ for V over k. Then we may also write $A = k\langle x_1, \dots, x_n\rangle/I$ where $k\langle x_1, \dots, x_n\rangle$ is the free algebra on $\{x_1, \dots, x_n\}$ over k.

In representation theory of algebras, Calabi-Yau algebras are important class of algebras to study.

Definition 1. A graded algebra S is called a d-dimensional Calabi-Yau if

(1) S has a resolution of finite length by finitely generated graded projective left S^e-modules, and

(2) for some $\ell \in \mathbb{Z}$, $\operatorname{Ext}^i_{S^e}(S, S^e) \cong \begin{cases} S(\ell) & \text{if } i = d, \\ 0 & \text{otherwise} \end{cases}$ as graded right S^e-modules.

An AS-regular algebra defined below is one of the first classes of algebras studied in noncommutative algebraic geometry. Since every noetherian graded Calabi-Yau algebra is AS-regular (see [11]), it is interesting to study such algebras from the point of view of both representation theory and noncommutative algebraic geometry.

Definition 2. A graded algebra S is called a d-dimensional AS-regular algebra if

(1) $\operatorname{gldim} S = d < \infty$, and

(2) for some $\ell \in \mathbb{Z}$, $\operatorname{Ext}^i_S(k, S) \cong \begin{cases} k(\ell) & \text{if } i = d, \\ 0 & \text{otherwise.} \end{cases}$

The detailed version of this paper has been submitted for publication elsewhere.

Calabi-Yau algebras are closely related to superpotentials. Let $\phi : V^{\otimes m} \to V^{\otimes m}$ be the m-cycle defined by

$$\phi(v_1 \otimes v_2 \otimes \cdots \otimes v_{m-1} \otimes v_m) := v_m \otimes v_1 \otimes \cdots \otimes v_{m-2} \otimes v_{m-1}.$$

Definition 3. Let $\mathsf{w} \in V^{\otimes m}$. We call w a superpotential if $\phi(\mathsf{w}) = \mathsf{w}$.

For a monomial $\mathsf{w} = x_{i_1} \cdots x_{i_m} \in k\langle x_1, \ldots, x_n\rangle_m \cong V^{\otimes m}$ of degree m, we define

$$\partial_{x_i}\mathsf{w} := \begin{cases} x_{i_2} \cdots x_{i_{m-1}} x_{i_m} & \text{if } i_1 = i \\ 0 & \text{if } i_1 \neq i. \end{cases}$$

We can extend ∂_{x_i} to the map $\partial_{x_i} : k\langle x_1, \ldots, x_n\rangle_m \to k\langle x_1, \ldots, x_n\rangle_{m-1}$ by linearity. We call ∂_{x_i} the cyclic derivative with respect to x_i.

Definition 4. Let $\mathsf{w} \in k\langle x_1, \ldots, x_n\rangle_m \cong V^{\otimes m}$ be a superpotential. Then the Jacobian algebra of w is the graded algebra of the form

$$J(\mathsf{w}) := k\langle x_1, \ldots, x_n\rangle/(\partial_{x_1}\mathsf{w}, \ldots, \partial_{x_n}\mathsf{w}).$$

We have the following theorem, which is a substantial part of the motivation for our study.

Theorem 5. (see [2], [3], [7] etc.) *If S is a 3-dimensional noetherian Calabi-Yau algebra generated in degree 1 over k, then there exists a unique superpotential w_S up to non-zero scalar multiples such that $S = J(\mathsf{w}_S)$.*

It is well-known that if S is a 3-dimensional noetherian Calabi-Yau algebra generated in degree 1 over k, then S is either 2-Koszul (quadratic) or 3-Koszul (cubic). Recently, Mori and Smith [8], [7] classified all superpotentials whose Jacobian algebras are 3-dimensional noetherian quadratic Calabi-Yau algebras, and computed the homological determinants of graded algebra automorphisms of such algebras. In this note, we investigate 3-dimensional noetherian cubic Calabi-Yau algebras using superpotentials. Note that our study is greatly related to Mori's article and Itaba's article in this proceedings. See also them.

2. Classification Results

It is known that if w is a superpotential such that $J(\mathsf{w})$ is 3-dimensional noetherian cubic Calabi-Yau, then $\mathsf{w} \in V^{\otimes 4}$ and $\dim_k V = 2$.

Example 6.
(1) Let $S = k\langle x, y\rangle/(xy^2 + y^2x, x^2y + yx^2)$. Then S a 3-dimensional noetherian cubic Calabi-Yau algebra. The corresponding superpotential w_S is given by

$$\mathsf{w}_S = x^2y^2 + yx^2y + y^2x^2 + xy^2x \in V^{\otimes 4} \quad (\dim_k V = 2).$$

(2) Now let $\dim_k V = 2$ and let $\mathsf{w} = xyxy + yxyx$. Then w is a superpotential in $V^{\otimes 4}$. However $J(\mathsf{w}) = k\langle x, y\rangle/(yxy, xyx)$ is not Calabi-Yau.

By the above example, it is interesting to know for which superpotential $\mathsf{w} \in V^{\otimes 4}$, $J(\mathsf{w})$ is a 3-dimensional noetherian cubic Calabi-Yau algebra. As the main result of this note, we give an explicit classification of 3-dimensional noetherian cubic Calabi-Yau superpotentials.

Theorem 7. ([10]) *Let* $\dim_k V = 2$. *The following is a classification of all noetherian cubic Calabi-Yau superpotentials* $0 \neq \mathsf{w} \in V^{\otimes 4}$ *up to isomorphism of* $J(\mathsf{w})$*:*

- w_1;
- $\mathsf{w}_1 + 2\mathsf{w}_2$;
- $\mathsf{w}_1 - 2\mathsf{w}_2$;
- $\alpha\mathsf{w}_1 + \beta\mathsf{w}_2 \ \ (\alpha \neq 0, \beta \neq 0, \alpha \neq \pm 2\beta)$;
- $\mathsf{w}_1 - 2\mathsf{w}_2 + \mathsf{w}_3$;
- $\mathsf{w}_1 + 4\mathsf{w}_5$;
- $\mathsf{w}_1 + 2\mathsf{w}_2 + 8\mathsf{w}_5$;
- $\mathsf{w}_1 - 2\mathsf{w}_2 - 2\mathsf{w}_5$;
- $\alpha\mathsf{w}_1 + \beta\mathsf{w}_2 + \mathsf{w}_5 \ \ (\alpha \neq 0, \beta \neq 0, \alpha \neq \pm 2\beta)$;
- $\alpha\mathsf{w}_1 + \mathsf{w}_5 + \mathsf{w}_6 \ \ (\alpha \neq 0, \pm\frac{1}{2})$;
- $\beta\mathsf{w}_2 + \mathsf{w}_5 + \mathsf{w}_6 \ \ (\beta \neq 0, \pm 1)$;
- $\alpha\mathsf{w}_1 + \beta\mathsf{w}_2 + \mathsf{w}_5 + \mathsf{w}_6 \ \ (\alpha \neq 0, \beta \neq 0, \pm 1, \beta \neq 2\alpha \pm 1, \beta \neq -2\alpha \pm 1)$;

where

$$\begin{aligned} \mathsf{w}_1 &= x^2y^2 + xy^2x + y^2x^2 + yx^2y, & \mathsf{w}_2 &= xyxy + yxyx, \\ \mathsf{w}_3 &= x^3y + x^2yx + xyx^2 + yx^3, & \mathsf{w}_4 &= y^3x + y^2xy + yxy^2 + xy^3, \\ \mathsf{w}_5 &= x^4, & \mathsf{w}_6 &= y^4. \end{aligned}$$

Using the above classification, we obtain the following result.

Theorem 8. ([10]) *Let* $\dim_k V = 2$ *and let* $0 \neq \mathsf{w} \in V^{\otimes 4}$ *be a superpotential. Then* $J(\mathsf{w})$ *is not 3-dimensional noetherian cubic Calabi-Yau if and only if* $J(\mathsf{w})$ *is isomorphic to one of the following five algebras:*

- $k\langle x, y\rangle/(x^3)$;
- $k\langle x, y\rangle/(x^3, x^2y, xyx, yx^2)$;
- $k\langle x, y\rangle/(yxy, xyx)$;
- $k\langle x, y\rangle/(yxy + x^3, xyx)$;
- $k\langle x, y\rangle/(x^3, y^3)$.

By [1], every 3-dimensional noetherian cubic Calabi-Yau algebra is a domain. On the other hand, the above five exceptional algebras are domains, so we have the following characterization.

Theorem 9. ([10]) *Let* $\dim_k V = 2$ *and let* $0 \neq \mathsf{w} \in V^{\otimes 4}$ *be a superpotential. Then* $J(\mathsf{w})$ *is 3-dimensional noetherian cubic Calabi-Yau if and only if it is a domain.*

Theorem 9 should be compared to the result in the quadratic case. In [8], Mori and Smith proved that if $\dim_k V = 3$ and $0 \neq \mathsf{w} \in V^{\otimes 3}$ is a superpotential, then $J(\mathsf{w})$ is 3-dimensional noetherian quadratic Calabi-Yau if and only if it is a domain.

Next we classify the point schemes of 3-dimensional noetherian cubic Calabi-Yau algebras. Before stating our result, we recall some basic facts about point schemes.

Definition 10. Let S be a 3-dimensional noetherian AS-regular algebra (generated in degree 1 over k). A graded S-module $M = \bigoplus_{i \in \mathbb{Z}} M_i$ is called a point module if

(1) M is cyclic with $M = M_0S$, and

(2) $\dim_k M_i = 1$ for all $i \geq 0$.

Theorem 11. (Artin, Tate and Van den Bergh [1]) *The point modules over a 3-dimensional noetherian AS-regular algebra S are parametrized by a scheme, called the point scheme. If E is the point scheme of a 3-dimensional noetherian quadratic AS-regular algebra, then E is $\mathbb{P}^2$ or a cubic curve in $\mathbb{P}^2$. If E is the point scheme of a 3-dimensional noetherian cubic AS-regular algebra, then E is $\mathbb{P}^1 \times \mathbb{P}^1$ or a bidegree $(2,2)$ curve in $\mathbb{P}^1 \times \mathbb{P}^1$.*

Furthermore, Artin, Tate and Van den Bergh [1] classified 3-dimensional noetherian AS-regular algebras in terms of their associated point schemes. Since then the point scheme plays an essential role in noncommutative algebraic geometry. We here show all possible point schemes for 3-dimensional noetherian cubic Calabi-Yau algebras.

Theorem 12. ([10]) *The point scheme E of any 3-dimensional noetherian cubic Calabi-Yau algebra is isomorphic to one of the following:*

- $\mathbb{P}^1 \times \mathbb{P}^1$;
- *the union of two curves of bidegree $(1,0)$ and two curves of bidegree $(0,1)$;*
- *the union of a double curve of bidegree $(1,0)$ and a double curve of bidegree $(0,1)$;*
- *the union of two curves of bidegree $(1,1)$ meeting at two points;*
- *the union of two curves of bidegree $(1,1)$ meeting at one point;*
- *a double curve of bidegree $(1,1)$;*
- *an irreducible curve of bidegree $(2,2)$ with a biflecnode;*
- *an irreducible curve of bidegree $(2,2)$ with a cusp;*
- *a smooth curve of bidegree $(2,2)$.*

Thus we see that not all bidegree $(2,2)$ curves in $\mathbb{P}^1 \times \mathbb{P}^1$ appear as point schemes.

Remark 13. This consequence is certainly different from the quadratic case. In [8], Mori and Smith proved that all cubic curves in $\mathbb{P}^2$ appear as point schemes of 3-dimensional noetherian quadratic Calabi-Yau algebras.

Example 14. Let $S = k\langle x, y\rangle/(xy^2 + y^2x + x^3, x^2y - yx^2)$. Then S is a 3-dimensional noetherian cubic AS-regular algebra whose point scheme is the union of curves of bidegree $(1,0), (0,1)$ and $(1,1)$. Since it is known that the point scheme is a graded Morita invariant, we see that S is a cubic AS-regular algebra which is not graded Morita equivalent to any Calabi-Yau algebra.

3. Homological Determinants

The homological determinant plays an important role in invariant theory for AS-regular algebras (see [4], [5], [6], [9] etc). In this section, we will compute the homological determinants for 3-dimensional noetherian cubic Calabi-Yau algebras.

Let A be a graded algebra, σ a graded algebra automorphism of A, and M, N graded right A-modules. A k-linear graded map $f : M \to N$ is called σ-linear if $f : M \to N_\sigma$ is a graded A-module homomorphism where M_σ is a graded right A-module defined by $M_\sigma = M$ as a graded vector space with the right action $m \circ a := m\sigma(a)$ for $m \in M$ and $a \in A$.

For a graded right A-module M, we define the i-th local cohomology of M by

$$\mathrm{H}^i_{\mathfrak{m}}(M) := \lim_{n\to\infty} \mathrm{Ext}^i_A(A/A_{\geq n}, M).$$

If S is a noetherian AS-regular algebra of dimension d, then $\mathrm{H}^d_{\mathfrak{m}}(S) \cong DS(-\ell)$ for some ℓ where $D(-) := \mathrm{Hom}_k(-,k)$. It follows from [5, Lemma 2.2] that a graded algebra automorphism σ induces a σ-linear map $\mathrm{H}^d_{\mathfrak{m}}(\sigma) : \mathrm{H}^d_{\mathfrak{m}}(S) \to \mathrm{H}^d_{\mathfrak{m}}(S)$. Moreover, there exists a constant $c \in k^\times$ such that $\mathrm{H}^d_{\mathfrak{m}}(\sigma) : \mathrm{H}^d_{\mathfrak{m}}(S) \to \mathrm{H}^d_{\mathfrak{m}}(S)$ is equal to $cD(\sigma^{-1}) : DS(\ell) \to DS(\ell)$. The constant c^{-1} is called the homological determinant of σ, and we denote $\mathrm{hdet}\,\sigma = c^{-1}$ (see [5, Definition 2.3]).

If $S = S(V) = k[x_1, \dots, x_n]$ is a commutative polynomial algebra and σ is a graded algebra automorphism of S, then $\mathrm{hdet}\,\sigma = \det \sigma|_V$ where det is the usual determinant.

The homological determinant is rather mysterious and difficult to compute from the definition. (In general, $\mathrm{hdet}\,\sigma \neq \det \sigma|_V$!) However, the following important theorem holds.

Theorem 15. ([5]) *If S is a noetherian AS-regular algebra of dimension d, and G is a finite group of graded algebra automorphisms of S such that* $\mathrm{hdet}\,\sigma = 1$ *for all $\sigma \in G$, then S^G is AS-Gorenstein of dimension d.*

Now let $S = T(V)/I = k\langle x,y\rangle/(xy^2 + yxy + y^2x, yx^2 + xyx + x^2y)$. Then S is a 3-dimensional noetherian cubic Calabi-Yau algebra. Let σ be the graded algebra automorphism defined by $\sigma(x) = x, \sigma(y) = -y$. Clearly $\det \sigma|_V = -1$. One can calculate that $\mathrm{hdet}\,\sigma = 1$, so $\mathrm{hdet}\,\sigma \neq \det \sigma|_V$. However, in this case, $\mathrm{hdet}\,\sigma = (\det \sigma|_V)^2$ holds. As a matter of fact, for some typical examples of 3-dimensional noetherian cubic Calabi-Yau algebras, it was shown that $\mathrm{hdet}\,\sigma = (\det \sigma|_V)^2$ holds for all σ. So we have the following question: for any 3-dimensional noetherian cubic Calabi-Yau algebra S and any graded algebra automorphism σ of S, does the formula $\mathrm{hdet}\,\sigma = (\det \sigma|_V)^2$ hold? To give an answer to the question, we describe the following useful theorem.

Theorem 16. ([7]) *Let S be a 3-dimensional noetherian cubic Calabi-Yau algebra and σ a graded algebra automorphism of S. Then $\sigma^{\otimes 4}(\mathsf{w}_S) = (\mathrm{hdet}\,\sigma)\mathsf{w}_S$.*

Using the above theorem and the classification in Theorem 7, we have a rather surprising result.

Theorem 17. ([10]) *Let S be a 3-dimensional noetherian cubic Calabi-Yau algebra. Then* $\mathrm{hdet}\,\sigma = (\det \sigma|_V)^2$ *holds for any graded algebra automorphism σ if and only if*

$$S \not\cong k\langle x,y\rangle/(xy^2 + yxy + y^2x + \sqrt{-3}x^3,\ yx^2 + xyx + x^2y + \sqrt{-3}y^3).$$

This result says that the answer to the question is positive with only one exceptional algebra. We can construct an explicit example which does not satisfy $\mathrm{hdet}\,\sigma = (\det \sigma|_V)^2$ as follows.

Example 18. Let $S = k\langle x,y\rangle/(xy^2 + yxy + y^2x + \sqrt{-3}x^3,\ yx^2 + xyx + x^2y + \sqrt{-3}y^3)$. Then we define the graded algebra automorphism σ of S by

$$\sigma(x) = x + \sqrt{-1}y,\ \sigma(y) = -x + \sqrt{-1}y.$$

We can check that $\det\sigma|_V = 2\sqrt{-1}$ and $\operatorname{hdet}\sigma = 2 - 2\sqrt{-3}$, so $\operatorname{hdet}\sigma \neq (\det\sigma|_V)^2$. However, in this case, $\operatorname{hdet}\sigma = \omega(\det\sigma|_V)^2$ holds where $\omega = \frac{-1+\sqrt{-3}}{2}$ is a primitive third root of unity.

References

[1] M. Artin, J. Tate and M. Van den Bergh, Some algebras associated to automorphisms of elliptic curves, The Grothendieck Festschrift, Vol. I, 33–85, *Progr. Math.*, **86**, Birkhäuser, Boston, MA, 1990.

[2] R. Bocklandt, Graded Calabi Yau algebras of dimension 3, *J. Pure Appl. Algebra* **212** (2008), 14–32.

[3] R. Bocklandt, T. Schedler and M. Wemyss, Superpotentials and higher order derivations, *J. Pure Appl. Algebra* **214** (2010), 1501–1522.

[4] K. Chan, E. Kirkman, C. Walton and J.J. Zhang, Quantum binary polyhedral groups and their actions on quantum planes, *J. Reine Angew. Math.* **719** (2016), 211–252.

[5] P. Jørgensen and J. J. Zhang, Gourmet's guide to Gorensteinness, *Adv. Math.* **151** (2000), 313–345.

[6] E. Kirkman and J. Kuzmanovich, Fixed subrings of Noetherian graded regular rings, *J. Algebra* **288** (2005), 463–484.

[7] I. Mori and S. P. Smith, m-Koszul Artin-Schelter regular algebras, *J. Algebra* **446** (2016), 373–399.

[8] I. Mori and S. P. Smith, The classification of Calabi-Yau algebras with 3 generators and 3 quadratic relations, preprint.

[9] I. Mori and K. Ueyama, Ample group actions on AS-regular algebras and noncommutative graded isolated singularities, *Trans. Amer. Math. Soc.* **368** (2016), 7359–7383.

[10] I. Mori and K. Ueyama, The classification of 3-dimensional noetherian cubic Calabi-Yau algebras, preprint, `arXiv:1606.00183`.

[11] M. Reyes, D. Rogalski and J. J. Zhang, Skew Calabi-Yau algebras and homological identities, *Adv. Math.* **264** (2014), 308–354.

Department of Mathematics
Faculty of Education
Hirosaki University
1 Bunkyocho, Hirosaki, Aomori 036-8560, JAPAN
E-mail address: `k-ueyama@hirosaki-u.ac.jp`