DECOMPOSITION THEORY OF MODULES THE CASE OF KRONECKER ALGEBRA

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Abstract. Let A be a finite-dimensional algebra over an algebraically closed field $\mathbb{k}$. For any finite-dimensional A-module M we give a general formula that computes the indecomposable decomposition of M without decomposing it. As an example we apply this formula to the Kronecker algebra A and give an explicit formula to compute the indecomposable decomposition of M, which enables us to make a computer program.

Key Words: Decomposition, Auslander-Reiten Theory, Topological Data Analysis, Kronecker Algebra, Quiver and Algebra.

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1. Introduction

1.1. **Introduction to Decomposition theory.** Throughout this paper $\mathbb{k}$ is an algebraically closed field, and all vector spaces, algebras and linear maps are assumed to be finite-dimensional $\mathbb{k}$-vector spaces, finite-dimensional $\mathbb{k}$-algebras and $\mathbb{k}$-linear maps, respectively. Further all modules over an algebra considered here are assumed to be finite-dimensional left modules.

Let A be an algebra. We denote by $\operatorname{mod} A$ the category of A-modules. And let $\mathcal{L}$ be a complete set of representatives of isoclasses of indecomposable A-modules. Then the Krull-Schmidt theorem states the following. For each A-module M, there exists a unique map $\boldsymbol{d}_M \colon \mathcal{L} \to \mathbb{N}_0$ such that

$$M \cong \bigoplus_{L \in \mathcal{L}} L^{(\boldsymbol{d}_M(L))},$$

which is called an *indecomposable decomposition* of M. Therefore, $M \cong N$ if and only if $\boldsymbol{d}_M = \boldsymbol{d}_N$ for all A-modules M and N, i.e., the map $\boldsymbol{d}_M$ is a complete invariant of M under isomorphisms. Note that since M is finite-dimensional, the support $\operatorname{supp}(\boldsymbol{d}_M) := \{L \in \mathcal{L} \mid \boldsymbol{d}_M(L) \neq 0\}$ of $\boldsymbol{d}_M$ is a finite set. We call such a theory a *decomposition theory* that computes the indecomposable decomposition of a module. Our purpose is to develope a decomposition theory by using the knowledge of AR-quivers. Thus in the case that $\mathcal{L}$ is already computed and all almost split sequences are known, we aim to compute

(I) $\boldsymbol{d}_M$ and
(II) a finite set S_M such that $\operatorname{supp}(\boldsymbol{d}_M) \subseteq S_M \subseteq \mathcal{L}$

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for all A-modules M. Note that (II) is needed to give a finite algorithm. If A is representation-finite (i.e., if the set $\mathcal{L}$ is finite), then the problem (II) is trivial because we can take $S_M := \mathcal{L}$.

In this paper, we will solve the problem (I) in the decomposition theory for any finite-dimensional algebra A. This turns out to be an extension of the result for $A = \Bbbk[x]$ below(see Theorem 1). In particular, for the Kronecker algebra $A = \Bbbk Q$ with $Q = (1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2)$, we will give an explicit formula for the problem (I) and a solution to the problem (II).

1.2. **Motivation.** This work is partially supported by JST CREST Mathematics "Topological data analysis for new descriptors on soft matters" (Research Director: Y. Hiraoka). In the topological data analysis, persistence modules M are modules over the path algebra $\Lambda_n = \Bbbk Q$, where Q is the quiver whose underlying graph is the Dynkin graph

$$1 \text{ —— } 2 \text{ —— } \cdots \text{ —— } n$$

of type A_n for a natural number n, or more generally, modules over an algebra of the form $\Lambda_m \otimes_{\Bbbk} \Lambda_n$, and its persistence diagram is nothing but $\boldsymbol{d}_M$, which plays a central role there. Our argument here can be applied to have a decomposition theory for persistence modules (See for instance papers [8], [4], [6]).

1.3. **Well-known result for the Jordan canonical form.** The decomposition theory for polynomial algebras in one variable $A = \Bbbk[x]$ is already well known. A finite-dimensional A-module is a pair (V, f) of a finite-dimensional $\Bbbk$-vector space V and an endomorphism f of V, and by fixing a basis of V we may regard $V = \Bbbk^d$ for $d := \dim V$ and f as a square matrix M of size d. In this way we identify (V, f) with M. In this case we may have $\mathcal{L} = \{J_i(\lambda) \mid i \geq 1, \lambda \in \Bbbk\}$, where $J_i(\lambda)$ is the Jordan cell of size $i \geq 1$ with eigenvalue $\lambda \in \Bbbk$. Let Λ be the set of all distinct eigenvalues of M and set $M_\lambda = M - \lambda E_d$ for $\lambda \in \Lambda$. Then the following is well known.

Theorem 1. *The problems* (I) *and* (II) *are solved as follows.*

A solution to (I)**:** *Let $i \in \mathbb{N}$ and $\lambda \in \Lambda$. Then*

$$\boldsymbol{d}_M(J_i(\lambda)) = \begin{cases} d + \operatorname{rank} M_\lambda^2 - 2 \operatorname{rank} M_\lambda & \text{if } i = 1; \text{ and} \\ \operatorname{rank} M_\lambda^{i+1} + \operatorname{rank} M_\lambda^{i-1} - 2 \operatorname{rank} M_\lambda^i & \text{if } i \geq 2; \end{cases} \tag{1.1}$$

A solution to (II)**:** $S_M = \{J_i(\lambda) \mid i \leq d, \lambda \in \Lambda\}$.

Note that it is also easy to give all almost split sequences over $\Bbbk[x]$. Namely, they are given as follows:

$$\begin{gathered} 0 \to J_1(\lambda) \to J_2(\lambda) \to J_1(\lambda) \to 0, \\ 0 \to J_i(\lambda) \to J_{i-1}(\lambda) \oplus J_{i+1}(\lambda) \to J_i(\lambda) \to 0 \end{gathered} \tag{1.2}$$

for all $i \geq 2$ and $\lambda \in \Bbbk$. The reader may notice a similarity between (1.1) and (1.2), which will be explained in Remark 6.

1.4. **Comparison with Dowbor-Mróz's work.** We are pointed out by Emerson Escolor that there was already a similar investigation [5] by Dowbor and Mróz in the literature, which we did not know before. Thus this work was done independently. We here list some relationships between their results and ours.

(1) They also have the same statement as Theorem 5 and its dual version, namely a solution to (I). Their proof is similar to the first version of ours using a "Cartan matrix" of the module category of an algebra A and an AR-matrix of A as its inverse, but the proof presented here does not use them and is much simplified by using the minimal projective resolutions of simple functors that are given by almost split sequences and sink maps into indecomposable injective modules (Proposition 4).

(2) To solve the problem (II) we used traces and rejects, which are easily computed and give us a decomposition of a module into the preprojective part, the preinjective part, the regular part with parameter ∞, and the regular part without parameter ∞. This together with Theorem 10 gives an effective computation of the indecomposable decomposition of a module M. For instance, if the preprojective part or the preinjective part of M is zero, it avoids unnecessary computations of the decomposition for those parts, in contrast, such computations are done in their algorithm repeatedly.

2. A general solution to Problem (I)

In this section we give a general solution to the problem (I) by using Auslander-Reiten theory for an arbitrary algebra A.

2.1. **A general solution to Problem (I).** Let L be an indecomposable A-module.

Definition 2. We set

$$\mathcal{S}_L := \operatorname{Hom}_A(L, \text{-})/\operatorname{rad}\operatorname{Hom}_A(L, \text{-}) : \operatorname{mod} A \to \operatorname{mod} \Bbbk.$$

It is well-known that $\mathcal{S}_L$ is a simple functor.

By definition of $\mathcal{S}_L$ we have $\mathcal{S}_L(X) \cong \Bbbk$ if $X \cong L$ and $\mathcal{S}_L(X) = 0$ if $X \ncong L$ for all indecomposable A-modules X. Therefore, the indecomposable decomposition $M \cong \bigoplus_{L\in\mathcal{L}} L^{(\boldsymbol{d}_M(L))}$ of M shows that

$$\mathcal{S}_L(M) \cong \Bbbk^{(\boldsymbol{d}_M(L))}.$$

This gives us the following.

Proposition 3. *Let M be an A-module. Then we have*

$$\boldsymbol{d}_M(L) = \dim \mathcal{S}_L(M).$$

Recall the following fundamental statement in the Auslander-Reiten theory (see Auslander-Reiten [3] or Assem-Simson-Skowroński [2, IV, 6.11.]):

Proposition 4. *Let L be an indecomposable A-module. When L is injective, let $f\colon L \to L/\operatorname{soc} L = \bigoplus_{X\in J_L} X^{(a(X))}$ be the canonical epimorphism (note that $J_L = \emptyset$ if L is simple injective). When L is non-injective, let $0 \to L \xrightarrow{f} \bigoplus_{X\in J_L} X^{(a(X))} \xrightarrow{g} \tau^{-1}L \to 0$ be an* *almost split sequence starting at L with $J_L \subseteq \mathcal{L}$ and $a(X) \geq 1$ ($X \in J_L$). Then the simple functor $\mathcal{S}_L$ has a minimal projective resolution*

$$0 \to \mathrm{Hom}_A(\tau^{-1}L, \text{-}) \xrightarrow{\mathrm{Hom}_A(g,\text{-})} \bigoplus_{X \in J_L} \mathrm{Hom}_A(X, \text{-})^{(a(X))} \xrightarrow{\mathrm{Hom}_A(f,\text{-})} \mathrm{Hom}_A(L, \text{-}) \xrightarrow{can} \mathcal{S}_L \to 0,$$

where $g = 0$ and $\tau^{-1}L = 0$ if L is injective.

This together with Proposition 3 readily gives us the following.

Theorem 5. *Let M be an A-module. Then we have*

$$\boldsymbol{d}_M(L) = \dim \mathrm{Hom}_A(L, M) - \sum_{X \in J_L} a(X) \dim \mathrm{Hom}_A(X, M) + \dim \mathrm{Hom}_A(\tau^{-1}L, M).$$

2.2. **Calculation of** $\mathrm{Hom}_A(X, Y)$**.** When an algebra A is of the form $\Bbbk Q/I$ for some quiver Q and some ideal I of $\Bbbk Q$, it is possible to compute $\dim \mathrm{Hom}_A(X, Y)$ for every $X, Y \in \mathrm{mod}\, A$ by using the rank of a suitable matrix as follows, and thus $\boldsymbol{d}_M(L)$ in Theorem 5 is computable. First regard A-modules X and Y as representations $(X(i), X(\alpha))_{i \in Q_0, \alpha \in Q_1}$ and $(Y(i), Y(\alpha))_{i \in Q_0, \alpha \in Q_1}$ of Q, respectively. Then by definition we have

$$\mathrm{Hom}_A(X, Y) = \{(f_i)_{i \in Q_0} \in \prod_{i \in Q_0} \mathrm{Hom}_{\Bbbk}(X(i), Y(i)) \ \mid Y(\alpha) f_i = f_j X(\alpha), \forall \alpha : i \to j \text{ in } Q_1\}.$$

Therefore

$$\mathrm{Hom}_A(X, Y) \cong \{\boldsymbol{x} \in \Bbbk^N \mid B\boldsymbol{x} = 0\},$$

where $N := \sum_{i \in Q_0} \dim X(i) \dim Y(i)$ and B is a $|Q_1| \times N$-matrix given as the coefficient matrix of the homogeneous system of linear equations $Y(\alpha) f_i - f_j X(\alpha) = 0$ for f_i. Hence we obtain the equality:

$$\dim \mathrm{Hom}_A(X, Y) = N - \mathrm{rank}\, B.$$

Remark 6. Let $A := \Bbbk[x]$ be the polynomial algebra in one variable, and $M = (\Bbbk^d, M)$ an A-module as in Theorem 1. Then we have

$$\dim \mathrm{Hom}_A(J_i(\lambda), M) = d - \mathrm{rank}\, M_\lambda^i,$$

which together with Theorem 5 and the formula (1.2) yields the formula (1.1).

3. The case of Kronecker algebra

3.1. **Kronecker algebra.** The Kronecker algebra A is a path algebra of the quiver $Q = (1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2)$, and the category $\mathrm{mod}\, A$ of finite-dimensional A-modules is equivalent to the category $\mathrm{rep}\, Q$ of finite-dimensional representations of Q over $\Bbbk$. We usually identify these categories. Recall that a representation M of Q is a diagram $M(1) \underset{M(\beta)}{\overset{M(\alpha)}{\rightrightarrows}} M(2)$ of vector spaces and linear maps, and the *dimension vector* of M is defined to be the pair $\underline{\dim}\, M := (\dim M(1), \dim M(2))$. When $\underline{\dim}\, M = (d_1, d_2)$, without loss of generality we may set $M(i) = \Bbbk^{d_i}$ for $i = 1, 2$ and $M(\alpha), M(\beta) \in \mathbb{M}_{d_2, d_1}(\Bbbk)$. We denote M by the pair of matrices $(M(\alpha), M(\beta))$.

We here list well known facts on the Kronecker algebra (see Ringel [7, 3.2] for instance).

Theorem 7. *For the Kronecker algebra A the following statements hold.*

(1) *The list $\mathcal{L}$ of indecomposables is given as follows.*

Preprojective indecomposables: $\mathcal{P} := \left\{ P_n := \left(\begin{bmatrix} E_{n-1} \\ {}^t\mathbf{0} \end{bmatrix}, \begin{bmatrix} {}^t\mathbf{0} \\ E_{n-1} \end{bmatrix} \right) \middle| \, n \geq 1 \right\}$,

Preinjective indecomposables: $\mathcal{I} := \{ I_n := ([E_{n-1}, \mathbf{0}], [\mathbf{0}, E_{n-1}]) \mid n \geq 1\}$,

Regular indecomposables:

$$\mathcal{R} := \{R_n(\lambda) := (E_n, J_n(\lambda)), R_n(\infty) := (J_n(0), E_n) \mid n \geq 1, \lambda \in \Bbbk\},$$

where $\mathbf{0}$ is the $n \times 1$ matrix with all entries 0. Note that

$$\underline{\dim}\, P_n = (n-1, n), \underline{\dim}\, I_n = (n, n-1), \underline{\dim}\, R_n(\lambda) = (n, n)$$

for all $n \in \mathbb{N}$ and $\lambda \in \mathbb{P}^1(\Bbbk) = \Bbbk \cup \{\infty\}$.

(2) *The Auslander-Reiten quiver (AR-quiver for short) of A has the following form:*

$$\begin{tikzcd}[column sep=small]
 & P_2 \arrow[dr, shift left] \arrow[dr, shift right] \arrow[rr, dashed] & & P_4 \arrow[r, dashed] & \cdots & & & \cdots \arrow[r, dashed] & I_3 \arrow[dr, shift left] \arrow[dr, shift right] \arrow[rr, dashed] & & I_1. \\
P_1 \arrow[ur, shift left] \arrow[ur, shift right] \arrow[rr, dashed] & & P_3 \arrow[ur, shift left] \arrow[ur, shift right] \arrow[r, dashed] & \cdots & & \boxed{\mathcal{R}} & \cdots \arrow[r, dashed] & I_4 \arrow[ur, shift left] \arrow[ur, shift right] \arrow[rr, dashed] & & I_2 \arrow[ur, shift left] \arrow[ur, shift right] &
\end{tikzcd}$$

In the above the rectangle part $\mathcal{R}$ is given as the disjoint union of a family $(\mathcal{R}_\lambda)_{\lambda \in \mathbb{P}^1(\Bbbk)}$ of "homogeneous tubes" $\mathcal{R}_\lambda$ that has the form

$$\begin{tikzcd}
R_1(\lambda) \arrow[r, shift left] \arrow[loop below, dotted] & R_2(\lambda) \arrow[l, shift left] \arrow[r, shift left] \arrow[loop below, dotted] & R_3(\lambda) \arrow[l, shift left] \arrow[r, shift left] \arrow[loop below, dotted] & \cdots \arrow[l, shift left]
\end{tikzcd}$$

where dotted loops mean that for all $n \in \mathbb{N}$ the Auslander-Reiten translation τ sends $R_n(\lambda)$ to itself: $\tau R_n(\lambda) = R_n(\lambda)$.

(3) *Let $X, Y \in \mathcal{L}$. If $\operatorname{Hom}_A(X, Y) \neq 0$, then X is "on the left" of Y, i.e., one of the following occurs:*
   (i) $X \cong P_m, Y \cong P_n$ *with* $m \leq n$,
   (ii) $X \in \mathcal{P}, Y \in \mathcal{R} \cup \mathcal{I}$,
   (iii) $X \cong R_m(\lambda), Y \cong R_n(\mu)$ *with* $\lambda = \mu$,
   (iv) $X \in \mathcal{R}, Y \in \mathcal{I}$, *or*
   (v) $X \cong I_m, Y \cong I_n$ *with* $m \geq n$.

3.2. **An explicit solution to Problem (I) for Kronecker algebra.** Throughout the rest of this paper A is the Kronecker algebra, and M is an A-module. To apply Theorem 5 we compute the dimensions of the spaces $\operatorname{Hom}_A(L, N)$ for all $L \in \mathcal{L}$ and $N \in \operatorname{mod} A$ following Subsection 2.2.

Definition 8. We first define the following matrices with $n \geq 1$, $\lambda \in \Bbbk$ (note that $P_1(M) = \mathsf{J}_{0,1}$ is an empty matrix).

$$P_n(M) := \left.\left[\overbrace{\begin{matrix} M(\beta) & M(\alpha) & 0 & 0 & \cdots & 0 \\ 0 & M(\beta) & M(\alpha) & 0 & \ddots & \vdots \\ 0 & 0 & M(\beta) & M(\alpha) & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & M(\beta) & M(\alpha) \end{matrix}}^{n \text{ blocks}} \right] \right\} n-1 \text{ blocks},$$

$$I_n(M) := \left.\overbrace{\begin{bmatrix} M(\beta) & 0 & 0 & \cdots & 0 \\ M(\alpha) & M(\beta) & 0 & \ddots & \vdots \\ 0 & M(\alpha) & M(\beta) & \ddots & 0 \\ 0 & 0 & M(\alpha) & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & M(\beta) \\ 0 & 0 & \cdots & 0 & M(\alpha) \end{bmatrix}}^{n \text{ blocks}}\right\} n+1 \text{ blocks},$$

$$R_n(\lambda, M) := \left.\overbrace{\begin{bmatrix} M_\lambda(\alpha,\beta) & 0 & 0 & \cdots & 0 \\ M(\alpha) & M_\lambda(\alpha,\beta) & 0 & \ddots & \vdots \\ 0 & M(\alpha) & M_\lambda(\alpha,\beta) & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & M(\alpha) & M_\lambda(\alpha,\beta) \end{bmatrix}}^{n \text{ blocks}}\right\} n \text{ blocks}, \textit{and}$$

$$R_n(\infty, M) := \left.\overbrace{\begin{bmatrix} M(\alpha) & 0 & 0 & \cdots & 0 \\ -M(\beta) & M(\alpha) & 0 & \ddots & \vdots \\ 0 & -M(\beta) & M(\alpha) & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -M(\beta) & M(\alpha) \end{bmatrix}}^{n \text{ blocks}}\right\} n \text{ blocks},$$

where we put $M_\lambda(\alpha, \beta) := \lambda M(\alpha) - M(\beta)$, and we define the following numbers.

$$p_1(M) := 0, p_n(M) := \operatorname{rank} P_n(M) \ (n \geq 2),$$
$$i_0(M) := 0, i_n(M) := \operatorname{rank} I_n(M) \ (n \geq 1),$$
$$r_n(\lambda, M) := \operatorname{rank} R_n(\lambda, M) \ (n \geq 1, \lambda \in \mathbb{P}^1(\Bbbk)).$$

Using the data above we can compute the dimensions of Hom spaces $\operatorname{Hom}_A(L, M)$ with L indecomposable as follows.

Proposition 9. *We have the following formulas:*

$$\dim \operatorname{Hom}_A(P_n, M) = \begin{cases} (n-1)d_1 - p_{n-1}(M) & (n \geq 2) \\ d_2 & (n = 1) \end{cases}$$
$$\dim \operatorname{Hom}_A(I_n, M) = nd_1 - i_n(M) \quad (n \geq 1)$$
$$\dim \operatorname{Hom}_A(R_n(\lambda), M) = nd_1 - r_n(\lambda, M) \quad (n \geq 1, \lambda \in \mathbb{P}^1(\Bbbk))$$

Proposition 9 and Theorem 5 give us a solution to the problem (I) for the Kronecker algebra as follows.

Theorem 10. *We have the following formulas:*

$$\boldsymbol{d}_M(P_n) = \begin{cases} 2p_n(M) - p_{n-1}(M) - p_{n+1}(M) & (n \geq 2), \\ d_2 - p_2(M)\ (= \dim \operatorname{Coker}[M(\beta)\ M(\alpha)]) & (n = 1), \end{cases}$$

$$\boldsymbol{d}_M(I_n) = \begin{cases} 2i_{n-1}(M) - i_{n-2}(M) - i_n(M) & (n \geq 2), \\ d_1 - i_1(M)\ (= \dim \operatorname{Ker}{}^t[M(\beta)\ M(\alpha)]) & (n = 1), \end{cases}$$

$$\boldsymbol{d}_M(R_n(\lambda)) = \begin{cases} r_{n-1}(\lambda, M) + r_{n+1}(\lambda, M) - 2r_n(\lambda, M) & (n \geq 2), \\ r_2(\lambda, M) - 2r_1(\lambda, M) & (n = 1). \end{cases}$$

Remark 11. For any $n \geq 2$, we have

$$\begin{array}{lcl} 2p_n(M) - p_{n-1}(M) - p_{n+1}(M) & = & (p_n(M) - p_{n-1}(M)) - (p_{n+1}(M) - p_n(M)), \\ 2i_{n-1}(M) - i_{n-2}(M) - i_n(M) & = & (i_{n-1}(M) - i_{n-2}(M)) - (i_n(M) - i_{n-1}(M)), \\ r_{n-1}(\lambda, M) + r_{n+1}(\lambda, M) - 2r_n(\lambda, M) & = & (r_{n+1}(\lambda, M) - r_n(\lambda, M) - (r_n(\lambda, M) - r_{n-1}(\lambda, M)). \end{array}$$

Namely, $\boldsymbol{d}_M$ is the absolute value of the second difference sequences of $\{p_n(M)\}, \{i_n(M)\}$ or $\{r_n(\lambda, M)\}$.

3.3. **An explicit solution to Problem (II) for Kronecker algebra.** Set $d := d_1 + d_2$. Let $F\colon \bigoplus_{L \in \mathcal{L}} L^{(\boldsymbol{d}_M(L))} \to M$ be an isomorphism. Then we have

$$M = P_M \oplus R_M \oplus I_M,$$

where P_M, R_M and I_M are the images of $\bigoplus_{L \in \mathcal{P}} L^{(\boldsymbol{d}_M(L))}$, $\bigoplus_{L \in \mathcal{R}} L^{(\boldsymbol{d}_M(L))}$ and $\bigoplus_{L \in \mathcal{I}} L^{(\boldsymbol{d}_M(L))}$ by F, respectively. To compute P_M, R_M and I_M we here use the trace and reject in a module of a class of modules (see Anderson–Fuller [1] for details). Let $\mathcal{U}$ be a class of modules in $\operatorname{mod} A$ and $M \in \operatorname{mod} A$. Recall that the *trace* $\mathsf{Tr}_M(\mathcal{U})$ of $\mathcal{U}$ in M and the *reject* $\mathsf{Rej}_M(\mathcal{U})$ of $\mathcal{U}$ in M are defined by

$$\mathsf{Tr}_M(\mathcal{U}) := \sum \{\operatorname{Im} f \mid f \in \operatorname{Hom}_A(U, M) \text{ for some } U \in \mathcal{U}\}, \text{and}$$

$$\mathsf{Rej}_M(\mathcal{U}) := \bigcap \{\operatorname{Ker} f \mid f \in \operatorname{Hom}_A(M, U) \text{ for some } U \in \mathcal{U}\}.$$

The following three steps give us a solution to Problem (II) for Kronecker algebra.
(Step 1) If $\{f_1, \ldots, f_a\}$ is a basis of $\operatorname{Hom}_A(M, P_d)$, then we have

$$\mathsf{Rej}_M(P_d) = \bigcap_{i=1}^{a} \operatorname{Ker} f_i = R_M \oplus I_M \quad \text{and hence} \quad P_M \cong M/\left(\bigcap_{i=1}^{a} \operatorname{Ker} f_i\right).$$

(Step 2) Similarly for (Step1), if $\{g_1, \ldots, g_b\}$ is a basis of $\operatorname{Hom}_A(I_d, R_M \oplus I_M)$, then

$$\mathsf{Tr}_{R_M \oplus I_M}(I_d) = \sum_{i=1}^{b} \operatorname{Im} g_i = I_M.$$

(Step 3) Since $R_M = (R_M(\alpha), R_M(\beta))$ is the direct sum of regular indecomposable modules, both $R_M(\alpha)$ and $R_M(\beta)$ are square matrices, say of size at most d. Put $R(\infty) := \mathsf{Tr}_{R_M}(R_d(\infty))$. Then $R_M = R(\infty) \oplus R'$ for some A-submodule $R' = (X', Y')$ of R_M such that R' has no direct summand of the form $R_n(\infty)$ for any n by Theorem

7(3)(iii). Since the matrix X' is invertible, we have $R' \cong (E_l, (X')^{-1}Y')$ for some $l \leq d$. Therefore, the set Λ of eigenvalues of $(X')^{-1}Y'$ is finite.

Then by the three steps above we obtain the following.

Theorem 12. *Set*

$$S_M := \{P_i, I_j, R_k(\lambda) \mid 1 \leq i \leq d, 1 \leq j \leq d, 1 \leq k \leq d, \lambda \in \Lambda \cup \{\infty\}\}.$$

Then this gives a solution to the problem (II) *for the Kronecker algebra.*

Note that we actually have a more precise solution to Problem (II).

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