ON MUTATION OF τ-TILTING MODULES

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ABSTRACT. Mutation of τ-tilting modules is a basic operation to construct a new support τ-tilting module from a given one by replacing a direct summand. The aim of this paper is to give a positive answer to the question posed in [2, Question 2.31] about mutation of τ-tilting modules.

Key Words: τ-tilting modules, silting complexes, mutation.

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1. INTRODUCTION

τ-tilting theory was introduced by Adachi, Iyama and Reiten [2] and completes (classical) tilting theory from the viewpoint of mutation. Note that τ-tilting theory has stimulated several investigations; in particular, there is a close relation between support τ-tilting modules (see definition 2.1 for details) and some other important notions in representation theory, such as torsion classes, silting complexes, cluster-tilting objects and $*$-modules (see [2, 3, 5, 9, 10, 12, 15] and so on). Since τ-tilting theory was introduced, many algebraists started to apply it to important classes of algebras (see [1, 7, 8, 11, 13, 14, 17] and so on).

Let us recall a main result in the paper [2]. Let Λ be a finite dimensional algebra and $T = X \oplus U$ a basic τ-tilting Λ-module with an indecomposable summand X satisfying $X \notin \operatorname{Fac} U$. Take an exact sequence

$$X \xrightarrow{f} U' \longrightarrow Y \longrightarrow 0 \tag{1.1}$$

with a minimal left $(\operatorname{add} U)$-approximation f. It is shown in [2, Theorem 2.30] that Y is either zero or a direct sum of copies of an indecomposable Λ-module Z, and we can obtain a new basic support τ-tilting Λ-module $\mu_X(T)$ called *mutation* of T with respect to X by $\mu_X(T) = U$ if $Y = 0$ and $\mu_X(T) = Z \oplus U$ if $Y \neq 0$. In fact, similar to the exact sequence (1.1) we can construct triangles for calculating mutation of silting objects. We have known that cones of triangle approximations are indecomposable(see Definition-Proposition 2.3 for details).

Naturally, they posed the following question.

Question 1.1. Assume that Y in (1.1) is nonzero. Is Y indecomposable?

A partial answer for the case when Λ is an endomorphism algebra of a cluster-tilting object was given by Yang and Zhu in [16, Corollary 4.17]. The aim of this paper is to give a positive answer to this question.

Theorem 1.2. If Y in (1.1) is nonzero, then it is indecomposable.

The detailed version of this paper will be submitted for publication elsewhere.

The idea of proof is to use the bijection between support τ-tilting modules and two-term silting complexes given in [2].

2. Proof of theorem

First we recall the definition of support τ-tilting modules from [2]. Denote by $|X|$ the number of non-isomorphic indecomposable direct summands of X for a Λ-module X.

Definition 2.1. Let $X \in \operatorname{mod}\Lambda$ and $P \in \operatorname{proj}\Lambda$.

(1) We call X *τ-rigid* if $\operatorname{Hom}_\Lambda(X, \tau X) = 0$. We call (X,P) a *τ-rigid pair* if X is τ-rigid and $\operatorname{Hom}_\Lambda(P, X)$=0.
(2) X is called *τ-tilting* if X is τ-rigid and $|X| = |\Lambda|$.
(3) X is called *support τ-tilting* if there exists an idempotent e of Λ such that X is a τ-tilting $(\Lambda/\langle e\rangle)$-module. We call (X,P) a *support τ-tilting pair* if (X,P) is τ-rigid and $|X| + |P| = |\Lambda|$.

Note that by [2, Proposition 2.3] (X,P) is a support τ-tilting pair for Λ if and only if X is a τ-tilting $\Lambda/\langle e\rangle$-module, where e is an idempotent of Λ such that $\operatorname{add} P = \operatorname{add}\Lambda e$. Moreover, if (X,P) and (X,P') are support τ-tilting pairs for Λ, then we have $\operatorname{add} P = \operatorname{add} P'$. Thus, any support τ-tilting Λ-module X can be extended to a support τ-tilting pair (X,P) uniquely and we can identify support τ-tilting modules from support τ-tilting pairs. We denote by $\mathrm{s}\tau\text{-tilt}\,\Lambda$ the set of isomorphism classes of basic support τ-tilting pairs (or equivalently, modules) for Λ.

Now we recall the definition of silting complexes from [3].

Definition 2.2. We call $P \in \mathrm{K}^{\mathrm{b}}(\operatorname{proj}\Lambda)$ *silting* if $\operatorname{Hom}_{\mathrm{K}^{\mathrm{b}}(\operatorname{proj}\Lambda)}(P,P[i])$=0 for any $i > 0$ and $\operatorname{thick} P= \mathrm{K}^{\mathrm{b}}(\operatorname{proj}\Lambda)$, where $\operatorname{thick} P$ is the smallest full subcategory of $\mathrm{K}^{\mathrm{b}}(\operatorname{proj}\Lambda)$ containing P and is closed under cones, $[\pm 1]$ and direct summands. A complex P=(P^i, d^i) in $\mathrm{K}^{\mathrm{b}}(\operatorname{proj}\Lambda)$ is called *two-term* if P^i=0 for all $i \neq 0, -1$.

We denote by $2\text{-silt}\,\Lambda$ the set of isomorphism classes of basic two-term silting complexes in $\mathrm{K}^{\mathrm{b}}(\operatorname{proj}\Lambda)$. Moreover, recall the definition of mutation of silting complexes from [3].

Definition-Proposition 2.3. [3, Theorem 2.31] Let $P = X \oplus Q$ be a basic silting complex in $\mathrm{K}^{\mathrm{b}}(\operatorname{proj}\Lambda)$ with an indecomposable summand X. We take a minimal left $(\operatorname{add} Q)$-approximation f and a triangle

$$X \xrightarrow{f} Q' \longrightarrow Y \longrightarrow X[1].$$

Then Y is indecomposable. The *left mutation* of P with respect to X is $\mu_X^-(P) := Y \oplus Q$. The *right mutation* is defined dually but we will not use it in this paper. Then the left mutation and the right mutation of P are also basic silting complexes.

The following result establishes a relation between $2\text{-silt}\,\Lambda$ and $\mathrm{s}\tau\text{-tilt}\,\Lambda$.

Theorem 2.4. [2, Theorem 3.2 and Corollary 3.9] There exists a bijection

$$2\text{-silt}\,\Lambda \longleftrightarrow \mathrm{s}\tau\text{-tilt}\,\Lambda \tag{2.1}$$

given by $2\text{-silt}\,\Lambda \ni P \mapsto H^0(P) \in s\tau\text{-tilt}\,\Lambda$ and $s\tau\text{-tilt}\,\Lambda \ni (M, P) \mapsto (P_1 \oplus P \overset{(f,0)}{\to} P_0) \in 2\text{-silt}\,\Lambda$, where $f : P_1 \to P_0$ is a minimal projective presentation of M. Moreover, the bijection (2.1) preserves mutation.

Now we give a proof of main result of this paper.

Proof of Theorem 1.2. Assume $Y \neq 0$ in (1.1).

By taking the minimal projective presentations of X and U. We have the following exact sequences:

$$P_X^{-1} \xrightarrow{d_X^{-1}} P_X^0 \xrightarrow{d_X^0} X \longrightarrow 0, \quad P_U^{-1} \xrightarrow{d_U^{-1}} P_U^0 \xrightarrow{d_U^0} U \longrightarrow 0.$$

Denote by $P_X = (P_X^{-1} \xrightarrow{d_X^{-1}} P_X^0)$ and $P_U = (P_U^{-1} \xrightarrow{d_U^{-1}} P_U^0)$. Then $P_T = P_X \oplus P_U$ gives a minimal projective presentation of T. Then by Theorem 2.4 it follows that P_T belongs to $2\text{-silt}\,\Lambda$ since T is a basic τ-tilting Λ-module. Also P_X is indecomposable since X is indecomposable.

Under above setting, $\mu_X(T) \in s\tau\text{-tilt}\,\Lambda$ and $\mu_{P_X}^-(P_T) \in 2\text{-silt}\,\Lambda$ correspond via the bijection (2.1) in Theorem 2.4. In particular, we have

$$\mu_X(T) = H^0(\mu_{P_X}^-(P_T)).$$

To calculate $\mu_{P_X}^-(P_T)$ in $\mathrm{K^b}(\mathrm{proj}\,\Lambda)$, we take a triangle

$$P_X \xrightarrow{a} P' \xrightarrow{b} Q \longrightarrow P_X[1], \tag{2.2}$$

where a is a minimal left $(\mathrm{add}\,P_U)$-approximation of P_X. Then by Definition-Proposition 2.3, Q is indecomposable and $\mu_{P_X}^-(P_T) = Q \oplus P_U$.

Denote by $H^0(-) : \mathrm{K^b}(\mathrm{proj}\,\Lambda) \to \mathrm{mod}\,\Lambda$ the 0th-cohomology functor. Taking the 0th-cohomology of the triangle (2.2), we obtain the following exact sequence:

$$X \xrightarrow{f'} U_{P'} \xrightarrow{g'} Y_Q \longrightarrow 0,$$

where $U_{P'} = H^0(P') \in H^0(\mathrm{add}\,P_U) = \mathrm{add}\,U$, $Y_Q = H^0(Q)$, $f' = H^0(a)$ and $g' = H^0(b)$. Since Q is indecomposable, it follows that Y_Q is indecomposable.

We claim that f' is a left $(\mathrm{add}\,U)$-approximation. For any $h \in \mathrm{Hom}_\Lambda(X, U)$, there exist morphisms c^{-1} and c^0 making the following diagram commutative:

$$\begin{tikzcd}
P_X^{-1} \arrow[r, "d_X^{-1}"] \arrow[d, "c^{-1}"] & P_X^0 \arrow[r, "d_X^0"] \arrow[d, "c^0"] & X \arrow[r] \arrow[d, "h"] & 0 \\
P_U^{-1} \arrow[r, "d_U^{-1}"'] & P_U^0 \arrow[r, "d_U^0"'] & U \arrow[r] & 0.
\end{tikzcd}$$

Define $c = \overline{(c^{-1}, c^0)} \in \mathrm{Hom}_{\mathrm{K^b}(\mathrm{proj}\,\Lambda)}(P_X, P_U)$. Immediately, we have $H^0(c) = h$.

Since $a \in \mathrm{Hom}_{\mathrm{K^b}(\mathrm{proj}\,\Lambda)}(P_X, P')$ is a left $(\mathrm{add}\,P_U)$-approximation, there exists $e \in \mathrm{Hom}_{\mathrm{K^b}(\mathrm{proj}\,\Lambda)}(P', P_U)$ such that $c = ea \in \mathrm{Hom}_{\mathrm{K^b}(\mathrm{proj}\,\Lambda)}(P_X, P_U)$. Since $H^0(-) : \mathrm{K^b}(\mathrm{proj}\,\Lambda) \to \mathrm{mod}\,\Lambda$ is a functor, we have

$$h = H^0(c) = H^0(ea) = H^0(e)H^0(a) = H^0(e)f'.$$

We have finished to prove that f' is a left $(\operatorname{add} U)$-approximation.

Since f is a minimal left $(\operatorname{add} U)$-approximation, there exists a module U'' in $\operatorname{add} U$ such that $U_{P'} = U' \oplus U''$ and $Y_Q = Y \oplus U''$. Since Y_Q is indecomposable and $Y \neq 0$ by our assumption, we have that $U'' = 0$ and $Y = Y_Q$ is indecomposable. $\square$

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