t-STRUCTURES AND SILTING OBJECTS

TAKAHIDE ADACHI

ABSTRACT. In this note, we study a relationship between bounded t-structures and silting objects. Keller-Vossieck showed that for the path algebra of a Dynkin quiver, there exists a bijection between the set of isoclasses of basic silting objects and the set of bounded t-structures. Unfortunately, it is known that a bounded t-structure is not necessarily given by a silting object. We give a characterization of a class of algebras which satisfies the condition that all bounded t-structures are given by silting objects.

Throughout this note, Λ is a finite dimensional algebra over a field. We denote by $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda)$ the bounded derived category of finitely generated Λ-modules and by $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$ the bounded homotopy category of finitely generated projective Λ-modules.

In this note, we study a relationship between bounded t-structures on $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda)$ and silting objects of $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$. Recall the definition of t-structures which are introduced by Beĭlinson-Bernstein-Deligne. For details, we refer to [2]. Let $\mathcal{T}$ be a triangulated category. A pair $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ of full subcategories of $\mathcal{T}$ is called a *t-structure* on $\mathcal{T}$ if the following conditions are satisfied:

(1) $\mathcal{T}^{\leq -1} \subset \mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq 1} \subset \mathcal{T}^{\geq 0}$.
(2) $\mathrm{Hom}(X, Y) = 0$ for all $X \in \mathcal{T}^{\leq 0}$ and $Y \in \mathcal{T}^{\geq 1}$.
(3) $\mathcal{T} = \mathcal{T}^{\leq 0} * \mathcal{T}^{\geq 1}$ (*i.e.*, for each object Z of $\mathcal{T}$, there exists a triangle $X \to Z \to Y \to X[1]$ with $X \in \mathcal{T}^{\leq 0}$ and $Y \in \mathcal{T}^{\geq 1}$).

Here, for each integer n, let $\mathcal{T}^{\leq n} := \mathcal{T}^{\leq 0}[-n]$ and $\mathcal{T}^{\geq n} := \mathcal{T}^{\geq 0}[-n]$.

We collect basic results of t-structures. Let $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ be a t-structure on $\mathcal{T}$. Note that, for each integer n, the pair $(\mathcal{T}^{\leq n}, \mathcal{T}^{\geq n})$ is also t-structure. The following statements hold.

(1) $\mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq 0}$ are additive subcategories which are closed under extensions and direct summands.
(2) The heart $\mathcal{T}^0 := \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$ is an abelian category.
(3) The inclusion $\mathcal{T}^{\leq 0} \to \mathcal{T}$ has a left adjoint functor $\sigma^{\leq 0}$ and the inclusion $\mathcal{T}^{\geq 0} \to \mathcal{T}$ has a right adjoint functor $\sigma^{\geq 0}$. Moreover, $\sigma^0 := \sigma^{\leq 0}\sigma^{\geq 0} : \mathcal{T} \to \mathcal{T}^0$ is a cohomological functor.

A t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is said to be *bounded* if

$$\mathcal{T} = \bigcup_{n \in \mathbb{Z}} \mathcal{T}^{\leq n} = \bigcup_{n \in \mathbb{Z}} \mathcal{T}^{\geq n},$$

or equivalently, if $\mathcal{T} = \mathsf{thick}(\mathcal{T}^0)$. It is called an *algebraic* t-structure if in addition the heart is a length category with finitely many nonisomorphic simple objects. We denote

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by t-str$\mathcal{T}$ the set of bounded t-structures on $\mathcal{T}$ and by t-str$_{alg}\mathcal{T}$ the subset of t-str$\mathcal{T}$ consisting of algebraic t-structures.

We give a well-known example of t-structures. Let Λ be a finite dimensional algebra. We define two subcategories of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\Lambda)$

$$\mathcal{D}_{\Lambda}^{\leq 0} := \{X \in \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\Lambda) \mid H^i(X) = 0 \text{ for each } i > 0\},$$
$$\mathcal{D}_{\Lambda}^{\geq 0} := \{X \in \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\Lambda) \mid H^i(X) = 0 \text{ for each } i < 0\},$$

where $H^i(X)$ is the i-th cohomology of X. Then $(\mathcal{D}_{\Lambda}^{\leq 0}, \mathcal{D}_{\Lambda}^{\geq 0})$ is a bounded t-structure on $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\Lambda)$. Moreover, it is algebraic because the heart is $\mathsf{mod}\Lambda$.

To study t-structures on $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\Lambda)$, Keller-Vossieck introduced the notion of silting objects which is a generalization of the notion of tilting objects. An object M of $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$ is said to be *silting* if $\mathrm{Hom}(M, M[i]) = 0$ for all integers $i > 0$, and $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda) = \mathsf{thick}M$.

Theorem 1. [8] *Let Λ be the path algebra of a Dynkin quiver. Then there exists a bijection between the set of isomorphism classes of basic silting objects of $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$ and the set of bounded t-structures on $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\Lambda)$.*

Recently, Koenig-Yang gave an analog of Theorem 1 for any finite dimensional algebra. For an object M, we define subcategories of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\Lambda)$ as follows:

$$\mathcal{D}_{M}^{\leq 0} := \{X \in \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\Lambda) \mid \mathrm{Hom}(M, X[i]) = 0 \text{ for each } i > 0\},$$
$$\mathcal{D}_{M}^{\geq 0} := \{X \in \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\Lambda) \mid \mathrm{Hom}(M, X[i]) = 0 \text{ for each } i < 0\}.$$

Recall that Λ is a silting object of $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$ and $(\mathcal{D}_{\Lambda}^{\leq 0}, \mathcal{D}_{\Lambda}^{\geq 0})$ is an algebraic t-structure on $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\Lambda)$. The correspondence is extended to the map from basic silting objects to algebraic t-structures. We denote by silt$\,\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$ the set of isomorphism classes of basic silting objects of $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$.

Theorem 2. [9] *Let Λ be a finite dimensional algebra. Then there exists a bijection*

$$\mathrm{silt}\,\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda) \to t\text{-str}_{alg}\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\Lambda)$$

given by $M \mapsto (\mathcal{D}_{M}^{\leq 0}, \mathcal{D}_{M}^{\geq 0})$. Moreover, the heart $\mathcal{D}_{M}^{0} := \mathcal{D}_{M}^{\leq 0} \cap \mathcal{D}_{M}^{\geq 0}$ is equivalent to $\mathsf{mod}\,\mathrm{End}(M)$.

From the viewpoint of the bijection above, Theorem 1 implies that, if Λ is the path algebra of a Dynkin quiver, then all bounded t-structures are algebraic. Our aim of this note is to show the following theorem, which is a generalization of Theorem 1. An algebra Λ is said to be *silting-discrete* if, for each integer $n > 0$, the set of isomorphism classes of basic n-term silting objects of $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$ is finite. Note that, for a silting object M, it is n-term if and only if it satisfies $\mathrm{Hom}(\Lambda, M[i]) = 0$ and $\mathrm{Hom}(M, \Lambda[i+n-1]) = 0$ for all integers $i > 0$.

Theorem 3. *Let Λ be a finite dimensional algebra. Then the following are equivalent:*

(a) *Λ is silting-discrete.*
(b) *All bounded t-structures on $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\Lambda)$ are algebraic.*

In the following, we give a sketch of the proof of Theorem 3. First, we show (a)$\Rightarrow$(b). The following result plays an important role. However, we skip the proof.

Proposition 4. *Let M be a basic silting object of $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$. Let $(\mathcal{D}^{\le 0}, \mathcal{D}^{\ge 0})$ be a bounded t-structure on $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\Lambda)$ satisfying $\mathcal{D}_M^{\le 0} \supsetneq \mathcal{D}^{\le 0}$. Then there exists a basic silting object N of $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$ such that $\mathcal{D}_M^{\le 0} \supsetneq \mathcal{D}_N^{\le 0} \supset \mathcal{D}^{\le 0}$.*

Proof of Theorem 3. (a)$\Rightarrow$(b): Assume that a bounded t-structure $(\mathcal{D}^{\le 0}, \mathcal{D}^{\ge 0})$ is not algebraic. We can easily check that there exists an integer $n > 0$ such that

$$\mathcal{D}_\Lambda^{\le 0} \supset \mathcal{D}^{\le 0} \supset \mathcal{D}_\Lambda^{\le -n+1}.$$

Since $(\mathcal{D}^{\le 0}, \mathcal{D}^{\ge 0})$ is not algebraic, we have $\mathcal{D}_\Lambda^{\le 0} \supsetneq \mathcal{D}^{\le 0}$. By Proposition 4, there exists a basic silting object M_1 such that

$$\mathcal{D}_\Lambda^{\le 0} \supsetneq \mathcal{D}_{M_1}^{\le 0} \supsetneq \mathcal{D}^{\le 0} \supsetneq \mathcal{D}_{\Lambda[n-1]}^{\le 0}.$$

Moreover, by applying Proposition 4, we have an infinite sequence

$$\mathcal{D}_\Lambda^{\le 0} \supsetneq \mathcal{D}_{M_1}^{\le 0} \supsetneq \mathcal{D}_{M_2}^{\le 0} \supsetneq \cdots \supsetneq \mathcal{D}_{M_k}^{\le 0} \supsetneq \cdots.$$

Then, for each silting object M_k, we obtain $\mathcal{D}_\Lambda^{\le 0} \supsetneq \mathcal{D}_{M_k}^{\le 0} \supsetneq \mathcal{D}_{\Lambda[n-1]}^{\le 0}$, and hence for each integer $i > 0$

$$\mathrm{Hom}(\Lambda, M_k[i]) = 0 \text{ and } \mathrm{Hom}(M_k, \Lambda[i+n-1]) = 0.$$

Namely, there exist infinitely many non-isomorphic basic n-term silting objects. This implies that Λ is not silting-discrete. $\square$

Next we show (b)$\Rightarrow$(a). We need the following result. A full subcategory $\mathcal{X}$ of $\mathsf{mod}\Lambda$ is called *torsion class* if it is closed under images and extensions. Moreover, it is called *functorially finite* if in addition there exists a Λ-module M such that $\mathcal{X} = \mathsf{Fac}(M)$.

Proposition 5. *Let Λ be a finite dimensional algebra. Then Λ is silting-discrete if and only if, for each basic silting object M, all torsion classes of $\mathsf{mod}\mathrm{End}(M)$ is functorially finite.*

Proof. By [1] and [7], an algebra Λ is silting-discrete if and only if, for each basic silting object M, the set f-tors$\mathrm{End}(M)$ of functorially finite torsion classes of $\mathsf{mod}\mathrm{End}(M)$ is finite. Moreover, by [4], the set f-tors$\mathrm{End}(M)$ is finite if and only if each torsion class of $\mathsf{mod}\mathrm{End}(M)$ is functorially finite. Hence the assertion follows. $\square$

Now we are ready to show Theorem 3.

Proof of Theorem 3. (b)$\Rightarrow$(a): By Proposition 5, we have only to show that, for each basic silting object M of $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$, all torsion classes of $\mathsf{mod}\mathrm{End}(M)$ are functorially finite. Indeed, let $\mathcal{X}$ be a torsion class of $\mathsf{mod}\mathrm{End}(M)$ and define a full subcategory

$$\mathcal{X}^{\perp} := \{Y \in \mathsf{mod}\mathrm{End}(M) \mid \mathrm{Hom}(X, Y) = 0 \text{ for each } X \in \mathcal{X}\}.$$

By [5], the pair $(\mathcal{D}_M^{\le -1} * \mathcal{X}, \mathcal{X}^{\perp}[1] * \mathcal{D}_M^{\ge 0})$ is also a bounded t-structure on $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\Lambda)$. Thus, by (b) and Theorem 2, there exists a basic silting object N of $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$ such that

$$\mathcal{D}_N^{\le 0} = \mathcal{D}_M^{\le -1} * \mathcal{X}.$$

On the other hand, since $\mathcal{D}_M^{\le 0} \supset \mathcal{D}_N^{\le 0} \supset \mathcal{D}_M^{\le -1}$ holds, we obtain

$$\mathcal{D}_N^{\le 0} = \mathcal{D}^{\le -1} * \mathcal{X}(N),$$

where $\mathcal{X}(N) := \mathsf{Fac}(\sigma_M^0(N))$ is a torsion class of $\mathsf{mod}\,\mathrm{End}(M)$ by [6, 3]. We can easily check that

$$\mathcal{X} = \mathcal{X}(N).$$

Hence, $\mathcal{X}$ is functorially finite. Therefore the assertion follows. $\square$

As a consequence of Theorem 3, a finite dimensional algebra Λ is silting-discrete if and only if the map $M \mapsto (\mathcal{D}_M^{\leq 0}, \mathcal{D}_M^{\geq 0})$ gives a bijection

$$\mathrm{silt}\,\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda) \to \text{t-str}\,\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda).$$

Since the path algebra of each Dynkin quiver is silting-discrete, we can recover Theorem 1 from our result.

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Faculty of Liberal Arts and Sciences
Osaka Prefecture University
1-1 Gakuen-cho, Nakaku, Sakai, Osaka, 599-8531, JAPAN
E-mail address: adachi@las.osakafu-u.ac.jp