Abstract. In this note, we study a relationship between bounded \( t \)-structures and silting objects. Keller-Vossieck showed that for the path algebra of a Dynkin quiver, there exists a bijection between the set of isoclasses of basic silting objects and the set of bounded \( t \)-structures. Unfortunately, it is known that a bounded \( t \)-structure is not necessarily given by a silting object. We give a characterization of a class of algebras which satisfies the condition that all bounded \( t \)-structures are given by silting objects.

Throughout this note, \( \Lambda \) is a finite dimensional algebra over a field. We denote by \( \text{D}^b(\text{mod}\Lambda) \) the bounded derived category of finitely generated \( \Lambda \)-modules and by \( \text{K}^b(\text{proj}\Lambda) \) the bounded homotopy category of finitely generated projective \( \Lambda \)-modules.

In this note, we study a relationship between bounded \( t \)-structures on \( \text{D}^b(\text{mod}\Lambda) \) and silting objects of \( \text{K}^b(\text{proj}\Lambda) \). Recall the definition of \( t \)-structures which are introduced by Beilinson-Bernstein-Deligne. For details, we refer to [2]. Let \( \mathcal{T} \) be a triangulated category. A pair \( (\mathcal{T}^\leq, \mathcal{T}^\geq) \) of full subcategories of \( \mathcal{T} \) is called a \( t \)-structure on \( \mathcal{T} \) if the following conditions are satisfied:

1. \( \mathcal{T}^\leq \cap \mathcal{T}^\geq = \mathcal{T}^\geq \cap \mathcal{T}^\leq \).
2. \( \text{Hom}(X, Y) = 0 \) for all \( X \in \mathcal{T}^\leq \) and \( Y \in \mathcal{T}^\geq \).
3. \( \mathcal{T} = \mathcal{T}^\leq \oplus \mathcal{T}^\geq \) (i.e., for each object \( Z \) of \( \mathcal{T} \), there exists a triangle \( X \to Z \to Y \to X[1] \) with \( X \in \mathcal{T}^\leq \) and \( Y \in \mathcal{T}^\geq \)).

Here, for each integer \( n \), let \( \mathcal{T}^\leq_n := \mathcal{T}^\leq[-n] \) and \( \mathcal{T}^\geq_n := \mathcal{T}^\geq[-n] \).

We collect basic results of \( t \)-structures. Let \( (\mathcal{T}^\leq, \mathcal{T}^\geq) \) be a \( t \)-structure on \( \mathcal{T} \). Note that, for each integer \( n \), the pair \( (\mathcal{T}^\leq_n, \mathcal{T}^\geq_n) \) is also \( t \)-structure. The following statements hold.

1. \( \mathcal{T}^\leq \) and \( \mathcal{T}^\geq \) are additive subcategories which are closed under extensions and direct summands.
2. The heart \( \mathcal{T}^0 := \mathcal{T}^\leq \cap \mathcal{T}^\geq \) is an abelian category.
3. The inclusion \( \mathcal{T}^\leq \to \mathcal{T} \) has a left adjoint functor \( \sigma^\leq \) and the inclusion \( \mathcal{T}^\geq \to \mathcal{T} \) has a right adjoint functor \( \sigma^\geq \). Moreover, \( \sigma^0 := \sigma^\leq \sigma^\geq : \mathcal{T} \to \mathcal{T}^0 \) is a cohomological functor.

A \( t \)-structure \( (\mathcal{T}^\leq, \mathcal{T}^\geq) \) is said to be **bounded** if

\[
\mathcal{T} = \bigcup_{n \in \mathbb{Z}} \mathcal{T}^\leq_n = \bigcup_{n \in \mathbb{Z}} \mathcal{T}^\geq_n,
\]

or equivalently, if \( \mathcal{T} = \text{thick}(\mathcal{T}^0) \). It is called an **algebraic** \( t \)-structure if in addition the heart is a length category with finitely many nonisomorphic simple objects. We denote

The detailed version of this paper will be submitted for publication elsewhere.
by \( t \)-str\( \mathcal{T} \) the set of bounded \( t \)-structures on \( \mathcal{T} \) and by \( t \)-str\_\text{alg} \( \mathcal{T} \) the subset of \( t \)-str\( \mathcal{T} \) consisting of algebraic \( t \)-structures.

We give a well-known example of \( t \)-structures. Let \( \Lambda \) be a finite dimensional algebra. We define two subcategories of \( \text{D}^b(\text{mod}\Lambda) \)
\[
\mathcal{D}_\Lambda^{\leq 0} := \{ X \in \text{D}^b(\text{mod}\Lambda) \mid H^i(X) = 0 \text{ for each } i > 0 \},
\]
\[
\mathcal{D}_\Lambda^{\geq 0} := \{ X \in \text{D}^b(\text{mod}\Lambda) \mid H^i(X) = 0 \text{ for each } i < 0 \},
\]
where \( H^i(X) \) is the \( i \)-th cohomology of \( X \). Then \( (\mathcal{D}_\Lambda^{\leq 0}, \mathcal{D}_\Lambda^{\geq 0}) \) is a bounded \( t \)-structure on \( \text{D}^b(\text{mod}\Lambda) \). Moreover, it is algebraic because the heart is \( \text{mod}\Lambda \).

To study \( t \)-structures on \( \text{D}^b(\text{mod}\Lambda) \), Keller-Vossieck introduced the notion of silting objects which is a generalization of the notion of tilting objects. An object \( M \) of \( \text{K}^b(\text{proj}\Lambda) \) is said to be silting if \( \text{Hom}(M, M[i]) = 0 \) for all integers \( i > 0 \), and \( \text{K}^b(\text{proj}\Lambda) = \text{thick} M \).

**Theorem 1.** [8] Let \( \Lambda \) be the path algebra of a Dynkin quiver. Then there exists a bijection between the set of isomorphism classes of basic silting objects of \( \text{K}^b(\text{proj}\Lambda) \) and the set of bounded \( t \)-structures on \( \text{D}^b(\text{mod}\Lambda) \).

Recently, Koenig-Yang gave an analog of Theorem 1 for any finite dimensional algebra. For an object \( M \), we define subcategories of \( \text{D}^b(\text{mod}\Lambda) \) as follows:
\[
\mathcal{D}_M^{\leq 0} := \{ X \in \text{D}^b(\text{mod}\Lambda) \mid \text{Hom}(M, X[i]) = 0 \text{ for each } i > 0 \},
\]
\[
\mathcal{D}_M^{\geq 0} := \{ X \in \text{D}^b(\text{mod}\Lambda) \mid \text{Hom}(M, X[i]) = 0 \text{ for each } i < 0 \}.
\]
Recall that \( \Lambda \) is a silting object of \( \text{K}^b(\text{proj}\Lambda) \) and \( (\mathcal{D}_\Lambda^{\leq 0}, \mathcal{D}_\Lambda^{\geq 0}) \) is an algebraic \( t \)-structure on \( \text{D}^b(\text{mod}\Lambda) \). The correspondence is extended to the map from basic silting objects to algebraic \( t \)-structures. We denote by \( \text{silt}\text{K}^b(\text{proj}\Lambda) \) the set of isomorphism classes of basic silting objects of \( \text{K}^b(\text{proj}\Lambda) \).

**Theorem 2.** [9] Let \( \Lambda \) be a finite dimensional algebra. Then there exists a bijection
\[
\text{silt}\text{K}^b(\text{proj}\Lambda) \rightarrow \text{t-str}_{\text{alg}} \text{D}^b(\text{mod}\Lambda)
\]
given by \( M \mapsto (\mathcal{D}_M^{\leq 0}, \mathcal{D}_M^{\geq 0}) \). Moreover, the heart \( \mathcal{D}_M^0 := \mathcal{D}_M^{\leq 0} \cap \mathcal{D}_M^{\geq 0} \) is equivalent to \( \text{modEnd}(M) \).

From the viewpoint of the bijection above, Theorem 1 implies that, if \( \Lambda \) is the path algebra of a Dynkin quiver, then all bounded \( t \)-structures are algebraic. Our aim of this note is to show the following theorem, which is a generalization of Theorem 1. An algebra \( \Lambda \) is said to be silting-discrete if, for each integer \( n > 0 \), the set of isomorphism classes of basic \( n \)-term silting objects of \( \text{K}^b(\text{proj}\Lambda) \) is finite. Note that, for a silting object \( M \), it is \( n \)-term if and only if it satisfies \( \text{Hom}(\Lambda, M[i]) = 0 \) and \( \text{Hom}(M, \Lambda[i+n-1]) = 0 \) for all integers \( i > 0 \).

**Theorem 3.** Let \( \Lambda \) be a finite dimensional algebra. Then the following are equivalent:

(a) \( \Lambda \) is silting-discrete.

(b) All bounded \( t \)-structures on \( \text{D}^b(\text{mod}\Lambda) \) are algebraic.

In the following, we give a sketch of the proof of Theorem 3. First, we show (a) \( \Rightarrow \) (b). The following result plays an important role. However, we skip the proof.
Proposition 4. Let $M$ be a basic silting object of $\text{K}^b(\text{proj}\Lambda)$. Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a bounded $t$-structure on $\text{D}^b(\text{mod}\Lambda)$ satisfying $\mathcal{D}^{\leq 0}_M \supset \mathcal{D}^{\leq 0}$. Then there exists a basic silting object $N$ of $\text{K}^b(\text{proj}\Lambda)$ such that $\mathcal{D}^{\leq 0}_N = \mathcal{D}^{\leq 0}_M$. 

Proof of Theorem 3. (a)⇒(b): Assume that a bounded $t$-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is not algebraic. We can easily check that there exists an integer $n > 0$ such that $\mathcal{D}^{\leq n}_A = \mathcal{D}^{\leq n+1}_A$. 

Since $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is not algebraic, we have $\mathcal{D}^{\leq 0} \supset \mathcal{D}^{\leq 0}_M$. By Proposition 4, there exists a basic silting object $M_1$ such that $\mathcal{D}^{\leq 0} \supset \mathcal{D}^{\leq 0}_M \supset \mathcal{D}^{\leq 0}_{M_1}$. 

Moreover, by applying Proposition 4, we have an infinite sequence $\mathcal{D}^{\leq 0}_A \supset \mathcal{D}^{\leq 0}_{M_1} \supset \mathcal{D}^{\leq 0}_{M_2} \supset \ldots \supset \mathcal{D}^{\leq 0}_{M_k} \supset \ldots$. 

Then, for each silting object $M_k$, we obtain $\mathcal{D}^{\leq 0}_A \supset \mathcal{D}^{\leq 0}_{M_k} \supset \mathcal{D}^{\leq 0}_{A[n-1]}$, and hence for each integer $i > 0$ $\text{Hom}(\Lambda, M_k[i]) = 0$ and $\text{Hom}(M_k, \Lambda[i + n - 1]) = 0$. 

Namely, there exist infinitely many non-isomorphic basic $n$-term silting objects. This implies that $\Lambda$ is not silting-discrete. 

Next we show (b)⇒(a). We need the following result. A full subcategory $\mathcal{X}$ of $\text{mod}\Lambda$ is called torsion class if it is closed under images and extensions. Moreover, it is called functorially finite if in addition there exists a $\Lambda$-module $M$ such that $\mathcal{X} = \text{Fac}(M)$. 

Proposition 5. Let $\Lambda$ be a finite dimensional algebra. Then $\Lambda$ is silting-discrete if and only if, for each basic silting object $M$, all torsion classes of $\text{modEnd}(M)$ is functorially finite. 

Proof. By [1] and [7], an algebra $\Lambda$ is silting-discrete if and only if, for each basic silting object $M$, the set f-torsEnd($M$) of functorially finite torsion classes of $\text{modEnd}(M)$ is finite. Moreover, by [4], the set f-torsEnd($M$) is finite if and only if each torsion class of $\text{modEnd}(M)$ is functorially finite. Hence the assertion follows. 

Now we are ready to show Theorem 3. 

Proof of Theorem 3. (b)⇒(a): By Proposition 5, we have only to show that, for each basic silting object $M$ of $\text{K}^b(\text{proj}\Lambda)$, all torsion classes of $\text{modEnd}(M)$ are functorially finite. Indeed, let $\mathcal{X}$ be a torsion class of $\text{modEnd}(M)$ and define a full subcategory $\mathcal{X}^\perp := \{Y \in \text{modEnd}(M) \mid \text{Hom}(X, Y) = 0 \text{ for each } X \in \mathcal{X}\}$. 

By [5], the pair $(\mathcal{D}^{\leq -1}_M \ast \mathcal{X}, \mathcal{X}^\perp [1] \ast \mathcal{D}^{\geq 0}_M)$ is also a bounded $t$-structure on $\text{D}^b(\text{mod}\Lambda)$. Thus, by (b) and Theorem 2, there exists a basic silting object $N$ of $\text{K}^b(\text{proj}\Lambda)$ such that $\mathcal{D}^{\leq 0}_N = \mathcal{D}^{\leq -1}_M \ast \mathcal{X}$.

On the other hand, since $\mathcal{D}^{\leq 0}_M \supset \mathcal{D}^{\leq 0}_N \supset \mathcal{D}^{\leq -1}_M$ holds, we obtain $\mathcal{D}^{\leq 0}_N = \mathcal{D}^{\leq -1}_M \ast \mathcal{X}(N)$.
where $\mathcal{X}(N) := \text{Fac}(\sigma_M^0(N))$ is a torsion class of $\text{modEnd}(M)$ by [6, 3]. We can easily check that

$$\mathcal{X} = \mathcal{X}(N).$$

Hence, $\mathcal{X}$ is functorially finite. Therefore the assertion follows. 

As a consequence of Theorem 3, a finite dimensional algebra $\Lambda$ is silting-discrete if and only if the map $M \mapsto (\mathcal{D}_M^{\leq 0}, \mathcal{D}_M^{> 0})$ gives a bijection

$$\text{siltKb}^b(\text{proj}\Lambda) \to \text{t-strD}^b(\text{mod}\Lambda).$$

Since the path algebra of each Dynkin quiver is silting-discrete, we can recover Theorem 1 from our result.

**References**


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