

BRICKS AND 2-TERM SIMPLE-MINDED COLLECTIONS

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ABSTRACT. Bricks are a generalization of simple modules, and they are fundamental in the representation theory of finite-dimensional algebras. We study sets of pairwise orthogonal bricks called semibricks in terms of τ -tilting theory. Our main results in this paper are canonical bijections between the set of support τ -tilting modules, the set of semibricks satisfying a certain condition called left-finiteness, and the set of 2-term simple-minded collections.

INTRODUCTION

The motivation of this paper is simple modules over a finite-dimensional algebra A over a field K . The following properties are well-known;

- for a simple A -module S , the endomorphism ring $\text{End}_A(S)$ is a division K -algebra,
- there is no nonzero map between nonisomorphic two simple A -modules,
- the smallest thick subcategory of the bounded derived category $\text{D}^b(\text{mod } A)$ containing all simple A -modules is $\text{D}^b(\text{mod } A)$ itself.

We consider objects having such properties in the module category $\text{mod } A$ (Section 1) and the derived category $\text{D}^b(\text{mod } A)$ (Section 2).

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NOTATIONS

Throughout of this paper, K is a field and A is a finite-dimensional K -algebra. The category of finite-dimensional right A -modules is denoted by $\text{mod } A$. Unless otherwise stated, algebras and modules are finite-dimensional, and subcategories are full subcategories.

For $M \in \text{mod } A$, the symbol $\text{ind } M$ denotes the set of isoclasses of indecomposable direct summands of M , and we set $|M| := \#(\text{ind } M)$. If $M \cong \bigoplus_{i=1}^m M_i^{n_i}$ with M_i indecomposable, $M_i \not\cong M_j$, and $n_i \geq 1$, then $\text{ind } M = \{M_1, \dots, M_m\}$ and $|M| = m$ hold.

1. SEMIBRICKS AND SUPPORT τ -TILTING MODULES

First, we define the new concept called semibricks.

Definition 1. We define as follows.

- (1) An A -module S in $\text{mod } A$ is called a *brick* if the endomorphism ring $\text{End}_A(S)$ is a division K -algebra.

The detailed version of this paper will be submitted for publication elsewhere. See [2].

- (2) A set \mathcal{S} of isoclasses of bricks in $\mathbf{mod} A$ is called a *semibrick* if $\mathrm{Hom}_A(S_1, S_2) = 0$ holds for any $S_1 \neq S_2 \in \mathcal{S}$. We define $\mathbf{sbrick} A$ as the set of semibricks in $\mathbf{mod} A$.

Some algebras, such as the Kronecker quiver algebra, admit semibricks consisting of infinitely many bricks. In such cases, there are so many semibricks in $\mathbf{mod} A$, but we investigate semibricks in terms of τ -tilting theory, so we only deal with the semibricks satisfying some condition on torsion pairs. The symbol $\mathbf{T}(\mathcal{S})$ (resp. $\mathbf{F}(\mathcal{S})$) denotes the smallest torsion (resp. torsion-free) class containing a semibrick \mathcal{S} , and the symbol $\mathbf{f-tors} A$ (resp. $\mathbf{f-torf} A$) denotes the set of functorially finite torsion (resp. torsion-free) classes.

Definition 2. Let $\mathcal{S} \in \mathbf{sbrick} A$. Then the semibrick \mathcal{S} is called *left finite* (resp. *right finite*) if $\mathbf{T}(\mathcal{S}) \in \mathbf{f-tors} A$ (resp. $\mathbf{F}(\mathcal{S}) \in \mathbf{f-torf} A$). We define $\mathbf{f}_L\text{-sbrick} A$ (resp. $\mathbf{f}_R\text{-sbrick} A$) as the subset of $\mathbf{sbrick} A$ consisting of all the left finite (resp. right finite) semibricks.

For example, a semibrick consisting of isoclasses of simple A -modules is left finite and right finite. We remark that a subset of a left finite semibrick is not necessarily left finite and that left finiteness is not generally equivalent to right finiteness.

Functorially finite torsion classes are strongly related with the support τ -tilting modules introduced by Adachi–Iyama–Reiten [1]. Let $M \in \mathbf{mod} A$. Then M is called a *support τ -tilting module* if there exists $P \in \mathbf{proj} A$ satisfying $\mathrm{Hom}_A(M, \tau M) = 0$, $\mathrm{Hom}_A(P, M) = 0$ and $|M| + |P| = |A|$. Here, $\mathbf{proj} A$ is the full subcategory of $\mathbf{mod} A$ consisting of the projective A -modules. If two projective modules P, Q satisfy the above conditions for a module M , then the additive closures $\mathbf{add} P, \mathbf{add} Q \subset \mathbf{mod} A$ coincide [1, Proposition 2.3]. Dually, the concept of support τ^{-1} -tilting modules is defined.

We define $\mathbf{s}\tau\text{-tilt} A$ (resp. $\mathbf{s}\tau^{-1}\text{-tilt} A$) as the set of isoclasses of basic support τ -tilting (resp. τ^{-1} -tilting) modules in $\mathbf{mod} A$. It is easy to see that $A, 0 \in \mathbf{s}\tau\text{-tilt} A$, and simple projective A -modules also belong to $\mathbf{s}\tau\text{-tilt} A$.

Adachi–Iyama–Reiten [1] obtained the following important result on support τ -tilting modules. For $M \in \mathbf{mod} A$, $\mathbf{Fac} M$ denotes the full subcategory of $\mathbf{mod} A$ consisting of factor modules of objects in $\mathbf{add} M$.

Proposition 3. [1, Theorem 2.7] *There exists a bijection $\mathbf{Fac}: \mathbf{s}\tau\text{-tilt} A \rightarrow \mathbf{f-tors} A$ sending $M \in \mathbf{s}\tau\text{-tilt} A$ to $\mathbf{Fac} M$.*

Now we state the main theorem of this section.

Theorem 4. *There exists a bijection $\mathbf{s}\tau\text{-tilt} A \rightarrow \mathbf{f}_L\text{-sbrick} A$ associating $M \in \mathbf{s}\tau\text{-tilt} A$ to $\mathbf{ind}(M/\mathbf{rad}_B M)$, where $B = \mathrm{End}_A(M)$.*

This map satisfies the following properties. The next proposition gives the way to calculate the corresponding left finite semibrick explicitly.

Proposition 5. *Let $M \in \mathbf{s}\tau\text{-tilt} A$ and $B := \mathrm{End}_A(M)$. Decompose M as $M = \bigoplus_{i=1}^m M_i$ with M_i indecomposable. We define*

$$L := \mathbf{rad}_B M, \quad N := M/L, \quad L_i := \sum_{f \in \mathbf{rad}_A(M, M_i)} \mathrm{Im} f \subset M_i, \quad N_i := M_i/L_i$$

for $i = 1, 2, \dots, m$. Then we have the following.

- (1) We have $L = \bigoplus_{i=1}^m L_i$ and $N = \bigoplus_{i=1}^m N_i$.

- (2) The module N_i is a brick or zero for each i .
- (3) The torsion class $\mathsf{T}(N)$ is equal to $\mathsf{Fac} M$.
- (4) We have $\mathsf{ind} N = \{N_i \mid N_i \neq 0\} \in \mathsf{f}_L\text{-sbrick } A$.

Especially, it follows that the cardinalities of left finite semibricks are bounded.

Corollary 6. *Every $\mathcal{S} \in \mathsf{f}_L\text{-sbrick } A$ has only $|A|$ elements at most.*

We obtain the relationship between the support τ -tilting modules, the left finite semibricks, the functorially finite torsion classes.

Proposition 7. *The map $\mathsf{T}: \mathsf{f}_L\text{-sbrick } A \rightarrow \mathsf{f-tors } A$ is a bijection, and we have the following commutative diagrams of bijections;*

$$\begin{array}{ccccc}
 \mathsf{s}\tau\text{-tilt } A & \xrightarrow{\mathsf{Fac}} & \mathsf{f-tors } A & \xleftarrow{\mathsf{T}} & \mathsf{f}_L\text{-sbrick } A \\
 & & & & \uparrow \\
 & & & & M \mapsto \mathsf{ind}(M/\mathsf{rad}_B M)
 \end{array}$$

We briefly explain mutations in $\mathsf{s}\tau\text{-tilt } A$. Let $M, N \in \mathsf{s}\tau\text{-tilt } A$ and take $P, Q \in \mathsf{proj } A$ satisfying the conditions of support τ -tilting modules for M, N , respectively. Then we say that N is a *mutation* of M if the pairs (M, P) and (N, Q) coincide except for exactly one indecomposable direct summand. For the detail, see [1, Definition 2.19]. By [1, Theorem 2.18], each direct summand of M or P admits a unique mutation. If N is a mutation of M in $\mathsf{s}\tau\text{-tilt } A$, then we have $\mathsf{Fac} M \supsetneq \mathsf{Fac} N$ or $\mathsf{Fac} M \subsetneq \mathsf{Fac} N$. In the first case, N is said to be a *left mutation* of M .

We have the following detail description of Theorem 4. The equivalence of (c) and (d) is already shown in [1, Definition-Proposition 2.28].

Proposition 8. *Let $M \in \mathsf{s}\tau\text{-tilt } A$ and $B := \mathsf{End}_A(M)$. Decompose M as $M = \bigoplus_{i=1}^m M_i$ with M_i indecomposable, and define L_i and N_i as in Proposition 5. Then the following conditions are equivalent for $i = 1, 2, \dots, m$.*

- (a) The module N_i is a brick.
- (b) The module N_i is nonzero.
- (c) The module M_i does not belong to $\mathsf{Fac} \bigoplus_{j \neq i} M_j$.
- (d) There exists a left mutation of M at M_i in $\mathsf{s}\tau\text{-tilt } A$.

Especially, the number of left mutations of M is equal to $|M/\mathsf{rad}_B M|$.

We define the exchange quiver of $\mathsf{s}\tau\text{-tilt } A$. The vertices are all the elements of $\mathsf{s}\tau\text{-tilt } A$, and for any two vertices M and N , there is an arrow from M to N if and only if N is a left mutation of M , and otherwise there is no arrow from M to N . For any vertex M with $B := \mathsf{End}_A(M)$, the number of arrows from M is $|M/\mathsf{rad}_B M|$, and the number of arrows to M is $|A| - |M/\mathsf{rad}_B M|$. We give an example.

Example 9. Let A be the path algebra of the quiver $1 \rightarrow 2 \rightarrow 3$. Figure 1 is the exchange quiver of $\mathsf{s}\tau\text{-tilt } A$. The bricks in $\mathsf{ind}(M/\mathsf{rad}_B M) \in \mathsf{f}_L\text{-sbrick } A$ for each $M \in \mathsf{s}\tau\text{-tilt } A$ are denoted by bold letters.

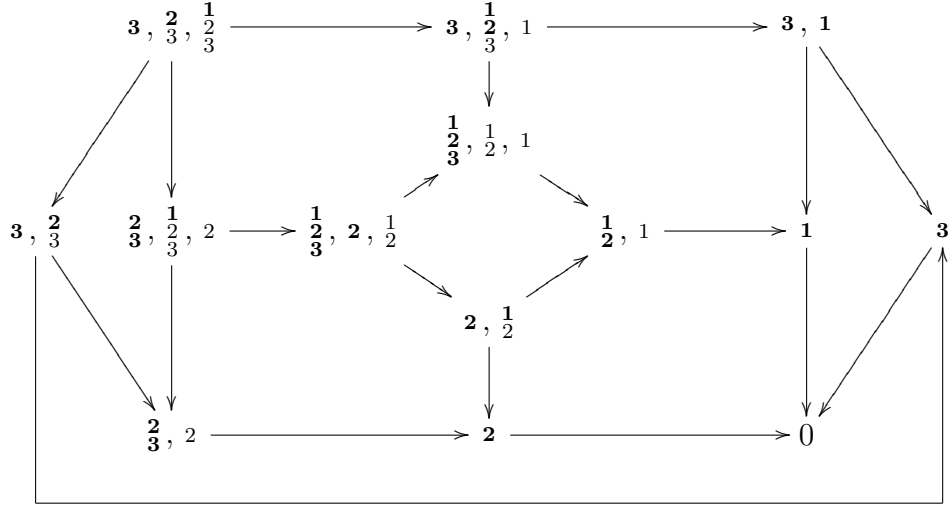


FIGURE 1. The exchange quiver of $s7$ -tilt A

2. SEMIBRICKS AND 2-TERM SIMPLE-MINDED COLLECTIONS

In this section, we study how left finite semibricks and right finite semibricks act in the bounded derived category $\mathbf{D}^b(\mathbf{mod} A)$. For this purpose, we first recall the definition of 2-term simple-minded collections in $\mathbf{D}^b(\mathbf{mod} A)$. Here, a full subcategory of a triangulated category is called *thick* if it is a triangulated subcategory closed under direct summands.

Definition 10. Let \mathcal{X} be a set of isomorphic classes of objects in $\mathbf{D}^b(\mathbf{mod} A)$. Then \mathcal{X} is called a *simple-minded collection* in $\mathbf{D}^b(\mathbf{mod} A)$ if it satisfies the following conditions;

- for any $X \in \mathcal{X}$, $\text{End}_{\mathbf{D}^b(\mathbf{mod} A)}(X)$ is a division K -algebra,
- for any $X_1 \neq X_2 \in \mathcal{X}$, we have $\text{Hom}_{\mathbf{D}^b(\mathbf{mod} A)}(X_1, X_2) = 0$,
- for any $X_1, X_2 \in \mathcal{X}$ and $n < 0$, we have $\text{Hom}_{\mathbf{D}^b(\mathbf{mod} A)}(X_1, X_2[n]) = 0$,
- the smallest thick subcategory of $\mathbf{D}^b(\mathbf{mod} A)$ containing \mathcal{X} is $\mathbf{D}^b(\mathbf{mod} A)$ itself.

Moreover, a simple-minded collection \mathcal{X} in $\mathbf{D}^b(\mathbf{mod} A)$ is said to be *2-term* if the i th cohomology $H^i(X)$ is 0 for any $i \neq -1, 0$ and any $X \in \mathcal{X}$. We write $2\text{-smc } A$ for the set of 2-term simple-minded collections in $\mathbf{D}^b(\mathbf{mod} A)$.

For example, the set of isoclasses of simple A -modules is a 2-term simple-minded collection in $\mathbf{D}^b(\mathbf{mod} A)$.

Every simple-minded collection \mathcal{X} in $\mathbf{D}^b(\mathbf{mod} A)$ has exactly $|A|$ elements [6, Lemma 3.3] even if it is not 2-term. If \mathcal{X} is 2-term, then every isoclass $X \in \mathcal{X}$ satisfies $X \in \mathbf{mod} A$ or $X \in (\mathbf{mod} A)[1]$, see [4, Remark 4.11].

Our first main result in this section is the following theorem.

Theorem 11. *There are bijections*

$$? \cap \mathbf{mod} A: 2\text{-smc } A \rightarrow \mathbf{f}_L\text{-sbrick } A, \quad ?[-1] \cap \mathbf{mod} A: 2\text{-smc } A \rightarrow \mathbf{f}_R\text{-sbrick } A$$

given by $\mathcal{X} \mapsto \mathcal{X} \cap \mathbf{mod} A$ and $\mathcal{X} \mapsto \mathcal{X}[-1] \cap \mathbf{mod} A$.

To understand the background of Theorem 11, we need some other concepts, and they play an important role in the proof of this theorem.

We first recall the definition of silting objects in the homotopy category $\mathbf{K}^b(\mathbf{proj} A)$.

An object $P \in \mathbf{K}^b(\mathbf{proj} A)$ is called a *silting object* in $\mathbf{K}^b(\mathbf{proj} A)$ if the following conditions are satisfied;

- for any $n > 0$, we have $\mathbf{Hom}_{\mathbf{K}^b(\mathbf{proj} A)}(P, P[n]) = 0$,
- the smallest thick subcategory of $\mathbf{K}^b(\mathbf{proj} A)$ containing P is $\mathbf{K}^b(\mathbf{proj} A)$ itself.

A silting object P in $\mathbf{K}^b(\mathbf{proj} A)$ is said to be *2-term* if it is isomorphic to some 2-term complex $(\cdots \rightarrow 0 \rightarrow P^{-1} \rightarrow P^0 \rightarrow 0 \rightarrow \cdots)$ in $\mathbf{K}^b(\mathbf{proj} A)$, where P^{-1} and P^0 are the -1 st and the 0th components, respectively. We write $2\text{-silt} A$ for the set of isoclasses of basic 2-term silting objects in $\mathbf{K}^b(\mathbf{proj} A)$. It is clear that A and $A[1]$ belong to $2\text{-silt} A$.

We dually define 2-term cosilting objects in $\mathbf{K}^b(\mathbf{inj} A)$ and a set $2\text{-cosilt} A$ as the set of isoclasses of basic ones.

Next we recall the definition of intermediate t-structures introduced by Beilinson–Bernstein–Deligne [3].

Let $(\mathcal{U}, \mathcal{V})$ be a pair of additive full subcategories of $\mathbf{D}^b(\mathbf{mod} A)$. The pair $(\mathcal{U}, \mathcal{V})$ is called a *t-structure* in $\mathbf{D}^b(\mathbf{mod} A)$ if it satisfies the following conditions;

- we have $\mathcal{U}[1] \subset \mathcal{U}$, $\mathcal{V}[-1] \subset \mathcal{V}$, and $\mathbf{Hom}_{\mathbf{D}^b(\mathbf{mod} A)}(\mathcal{U}, \mathcal{V}[-1]) = 0$,
- for every $X \in \mathbf{D}^b(\mathbf{mod} A)$, there exists an exact triangle $U \rightarrow X \rightarrow V \rightarrow Y[1]$ in $\mathbf{D}^b(\mathbf{mod} A)$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}[-1]$.

For a t-structure $(\mathcal{U}, \mathcal{V})$ in $\mathbf{D}^b(\mathbf{mod} A)$, a full subcategory $\mathcal{U} \cap \mathcal{V}$ called the *heart* is an abelian category [3, Théorème 1.3.6]. Sometimes t-structures with length heart (that is, their hearts are length categories) are easier to deal with than the other ones, and we will consider only such t-structures in this paper.

A typical example of t-structures in $\mathbf{D}^b(\mathbf{mod} A)$ is given as the *standard t-structure* $(\mathcal{U}_{\text{std}}, \mathcal{V}_{\text{std}})$ defined by cohomologies. It is explicitly written as follows;

$$\begin{aligned}\mathcal{U}_{\text{std}} &= \{X \in \mathbf{D}^b(\mathbf{mod} A) \mid H^i(X) = 0 \ (i > 0)\}, \\ \mathcal{V}_{\text{std}} &= \{X \in \mathbf{D}^b(\mathbf{mod} A) \mid H^i(X) = 0 \ (i < 0)\}.\end{aligned}$$

We say a t-structure $(\mathcal{U}, \mathcal{V})$ in $\mathbf{D}^b(\mathbf{mod} A)$ is *intermediate* with respect to the standard t-structure (or simply *intermediate*) if $\mathcal{U}_{\text{std}}[1] \subset \mathcal{U} \subset \mathcal{U}_{\text{std}}$ holds. Note that it is equivalent to $\mathcal{V}_{\text{std}}[1] \supset \mathcal{V} \supset \mathcal{V}_{\text{std}}$. We define a set $\text{int-t-str} A$ as the set of intermediate t-structures in $\mathbf{D}^b(\mathbf{mod} A)$ with length heart.

Koenig–Yang [6] put together the canonical bijections between the set of simple-minded collections in $\mathbf{D}^b(\mathbf{mod} A)$, the set of silting objects in $\mathbf{K}^b(\mathbf{proj} A)$, and the set of t-structures in $\mathbf{D}^b(\mathbf{mod} A)$ with length heart. Brüstle–Yang [4] showed that these bijections are restricted to the corresponding “2-term” concepts.

Proposition 12. [4, Corollary 4.3] *We have the following bijections.*

- (1) *There is a bijection $2\text{-silt} A \rightarrow \text{int-t-str} A$ given by $P \mapsto (P[>0]^\perp, P[<0]^\perp)$, where*

$$\begin{aligned}P[>0]^\perp &= \{X \in \mathbf{D}^b(\mathbf{mod} A) \mid \mathbf{Hom}_{\mathbf{D}^b(\mathbf{mod} A)}(P, X[n]) = 0 \ (n > 0)\}, \\ P[<0]^\perp &= \{X \in \mathbf{D}^b(\mathbf{mod} A) \mid \mathbf{Hom}_{\mathbf{D}^b(\mathbf{mod} A)}(P, X[n]) = 0 \ (n < 0)\}.\end{aligned}$$

- (2) *There exists a bijection $\text{int-t-str } A \rightarrow 2\text{-smc } A$ given as follows; each $(\mathcal{U}, \mathcal{V}) \in \text{int-t-str } A$ is sent to the set of isoclasses of simple objects in the heart $\mathcal{U} \cap \mathcal{V}$.*

We also need the following proposition to relate concepts in $D^b(\text{mod } A)$ or $K^b(\text{proj } A)$ with ones in $\text{mod } A$.

Proposition 13. *We have the following bijections.*

- (1) [1, Theorem 3.2] *The map $2\text{-silt } A \ni P \mapsto H^0(P) \in \text{s}\tau\text{-tilt } A$ is bijective.*
(2) [4, Theorem 4.9] *We have a bijection $(\text{heart}) \cap \text{mod } A: \text{int-t-str } A \rightarrow \text{f-tors } A$ defined by $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}) \mapsto \mathcal{H} \cap \text{mod } A$, where $\mathcal{H} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is the heart. Moreover, this bijection joins the following commutative diagram of bijections;*

$$\begin{array}{ccccc}
2\text{-silt } A & \xrightarrow{P \mapsto (P[>0]^\perp, P[<0]^\perp)} & \text{int-t-str } A & \xrightarrow{\text{simples of heart}} & 2\text{-smc } A \\
\downarrow H^0 & & \downarrow (\text{heart}) \cap \text{mod } A & & \\
\text{s}\tau\text{-tilt } A & \xrightarrow{\text{Fac}} & \text{f-tors } A & &
\end{array}$$

With these propositions, Theorem 11 is proved.

We finally obtain our second main theorem in this section. Figure 2 shows some of the known bijections and our new bijections (arrows with labels in rectangles).

Theorem 14. *We have the following assertions.*

- (1) *The diagram in Figure 2 is commutative and all the maps are bijective. In this diagram, $\mathcal{T} \in \text{f-tors } A$ corresponds to $\mathcal{F} \in \text{f-torf } A$ if and only if $(\mathcal{T}, \mathcal{F})$ is a torsion pair in $\text{mod } A$.*
(2) *If $\mathcal{X} \in 2\text{-smc } A$ corresponds to $\mathcal{S} \in \text{f}_L\text{-sbrick } A$ and $\mathcal{S}' \in \text{f}_R\text{-sbrick } A$, then we have $\mathcal{X} = \mathcal{S} \cup \mathcal{S}'[1]$ and a torsion pair $(\text{T}(\mathcal{S}), \text{F}(\mathcal{S}'))$ in $\text{mod } A$.*

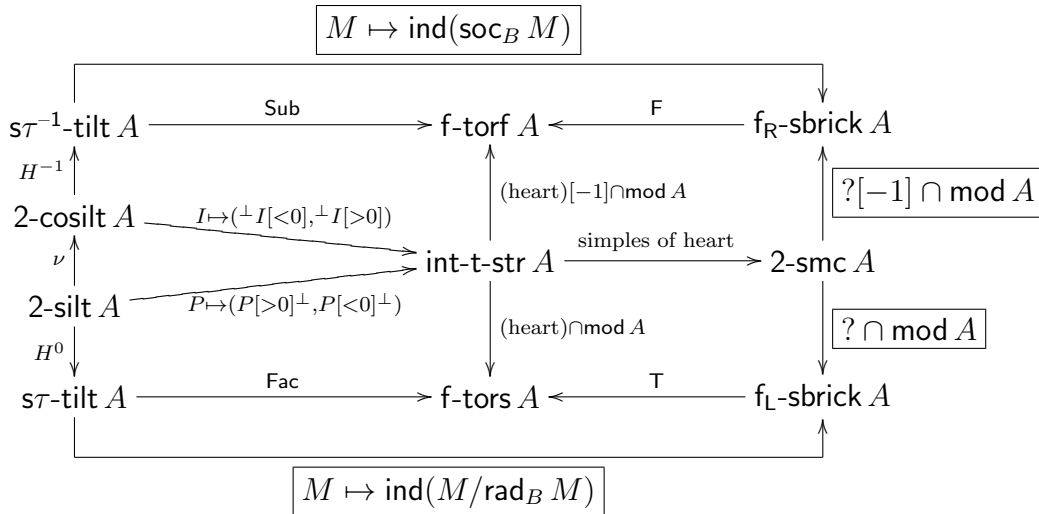


FIGURE 2. The commutative diagram

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