DERIVED EQUIVALENCES AND SMASH PRODUCTS

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Abstract. Let $\Bbbk$ and G be a commutative ring and a group, respectively. In the paper [1] (a final form in [2]) we investigated when the orbit categories of a pair of derived equivalent small $\Bbbk$-categories with G-actions are derived equivalent. Here we solve the converse problem, which is equivalent to the following problem: Let A and B be G-graded $\Bbbk$-categories, and assume that they are derived equivalent. Then under which condition are the smash products $A\#G$ and $B\#G$ derived equivalent?

Key Words: derived categories, derived equivalences, orbit categories, smash products.

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1. Introduction

Throughout this note $\Bbbk$ is a commutative ring and G is a group. A covering theory for derived equivalences was developed and applied for derived equivalece classifications of some classes of algebras such as representation-finite self-injective algebras or twisted multifold extensions of piecewise hereditary algebras of tree type. The main tool (in a generalized form) is given by the following theorem:

Theorem 1. *Let $(\mathcal{C}, X), (\mathcal{C}', X')$ be $\Bbbk$-categories with G-actions. Consider the following conditions.*

- (1′) *There exists a tilting subcategory $\mathcal{U}$ of $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}(\mathcal{C}, X))$ with a G-action Y such that the inclusion $(\mathcal{U}, Y) \hookrightarrow \mathcal{K}^{\mathrm{b}}(\mathrm{prj}(\mathcal{C}, X))$ is extended to a G-equivariant functor and $(\mathcal{C}', X')$ and $(\mathcal{U}, Y)$ are G-equivariantly equivalent.*
- (2) *The orbit categories $\mathcal{C}/G$ and $\mathcal{C}'/G$ are derived equivalent.*

Then the condition (1′) *implies* (2).

In the above a $\Bbbk$-category $\mathcal{C}$ with a G-action $X\colon G \to \mathrm{Aut}(\mathcal{C})$, $a \mapsto X_a$ is denoted by $(\mathcal{C}, X)$ and $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}(\mathcal{C}, X))$ denotes the bounded homotopy category of finitely generated projective $\mathcal{C}$-modules, which is a triangulated $\Bbbk$-category with a canonical G-action induced by X. Moreover, if $(\mathcal{D}, S)$ and $(\mathcal{E}, T)$ are $\Bbbk$-categories with G-actions, then a G-equivariant functor from $(\mathcal{D}, U)$ to $(\mathcal{E}, W)$ is a pair (F, ϕ) of a functor $F\colon \mathcal{D} \to \mathcal{E}$ and

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a family $\phi = (\phi_a)_{a\in G}$ of natural isomorphisms $\phi_a\colon FU_a \Longrightarrow V_aF$ making the diagram

$$\begin{tikzcd} FU_{ba} \arrow[rr, Rightarrow, "\phi_{ba}"] \arrow[d, equal] & & V_{ba}F \arrow[d, equal] \\ FU_bU_a \arrow[r, Rightarrow, "\phi_b U_a"] & V_bFU_a \arrow[r, Rightarrow, "V_b\phi_a"] & V_bB_aF \end{tikzcd}$$

commutative for all $a, b \in G$, and here (F, ϕ) is called a *G-equivarant equivalence* if F is an equivalence. Finally $(\mathcal{D}, S)$ and $(\mathcal{E}, T)$ are said to be *G-equivariantly equivalent* if there exists a G-equivarantly equivalence $(\mathcal{D}, S) \to (\mathcal{E}, T)$.

In this note we examine the converse problem:

Problem 1. For $\Bbbk$-categories $\mathcal{C}, \mathcal{C}'$ with G-actions, under which condition does a derived equivalence between $\mathcal{C}/G$ and $\mathcal{C}'/G$ yield a derived equivalence between $\mathcal{C}$ and $\mathcal{C}'$?

Here we recall the following theorem in [3].

Theorem 2. *The orbit category construction and the smash product construction are extended to 2-equivalences* $(\text{-})/G\colon G\text{-}\mathbf{Cat} \to G\text{-}\mathbf{GrCat}$ *and* $(\text{-})\#G\colon G\text{-}\mathbf{GrCat} \to G\text{-}\mathbf{Cat}$, *respectively, and they are 2-quasi-inveses to each other.*

By this theorem Problem 1 is reduced to the following.

Problem 1′. For G-graded $\Bbbk$-categories $\mathcal{B}, \mathcal{B}'$, under which condition does a derived equivalence between $\mathcal{B}$ and $\mathcal{B}'$ yield a derived equivalence between $\mathcal{B}\#G$ and $\mathcal{B}'\#G$?

Indeed, if this is known, then the obtained condition on $\mathcal{B} := \mathcal{C}/G$ and $\mathcal{B}' := \mathcal{C}'/G$ gives us an answer to Problem 1 because under this condition a deried equivalence between $\mathcal{C}/G$ and $\mathcal{C}'/G$ yields a derived equivalence between $(\mathcal{C}/G)\#G$ $(\simeq \mathcal{C})$ and $(\mathcal{C}'/G)\#G$ $(\simeq \mathcal{C}')$.

Recall the following theorem due to Rickard (the algebra case) and Keller (the category case) [even in the category case it is known that by a recent observation in [4] the $\Bbbk$-flatness condition on the category $\mathcal{C}'$ is not necessary any more]:

Theorem 3. *Let $\mathcal{C}$ and $\mathcal{C}'$ be $\Bbbk$-categories. Then the following are equivalent:*

(1) *$\mathcal{C}$ and $\mathcal{C}'$ are derived equivalent.*

(1″) *There exists a tilting subcategory $\mathcal{U}$ of $\mathcal{K}^{\mathrm{b}}(\operatorname{prj}\mathcal{C})$ such that $\mathcal{C}'$ and $\mathcal{U}$ are equivalent.*

By this theorem we may write $(1') \Leftrightarrow (1) \wedge (\alpha)$ for some extra condition (α) on G-actions. Precisely speaking, our purpose is to know a condition (β) on G-gradings on $\mathcal{C}/G$ and $\mathcal{C}'/G$ such that $(1) \wedge (\alpha) \Leftrightarrow (2) \wedge (\beta)$.

Throughout the rest of this paper all categories and functors are assumed to be $\Bbbk$-linear, and we call a category with a G-action a G-category for short. For a category $\mathcal{C}$ we denote by $\operatorname{prj}\mathcal{C}$ and $\operatorname{mod}\mathcal{C}$ the category of finitely generated $\mathcal{C}$-modules and the category of finitely generated $\mathcal{C}$-modules. By $\mathcal{C}^{\mathrm{b}}(\operatorname{prj}\mathcal{C})$ we denote the category of bounded complexes in $\operatorname{prj}\mathcal{C}$ and its homotopy category is denoted by $\mathcal{K}^{\mathrm{b}}(\operatorname{prj}\mathcal{C})$.

2. The 2-category G-$\mathbf{GrCat}$ and equivalences in it

Definition 4. G-$\mathbf{GrCat}$ is a 2-category defined as follows.

Objects: $(G\text{-}\mathbf{GrCat})_0$ is the class of G-graded small categories.

1-morphims: Let $\mathcal{B}, \mathcal{A} \in (G\text{-}\mathbf{GrCat})_0$. Then

$$G\text{-}\mathbf{GrCat}(\mathcal{B}, \mathcal{A}) := \{(H, r) \mid H\colon \mathcal{B} \to \mathcal{A} \text{ is a functor}, r\colon \mathcal{B}_0 \to G \text{ is a map with } H(\mathcal{B}^a(x,y)) \subseteq \mathcal{A}^{r(y)^{-1}ar(x)}(Hx, Hy)\ (x, y \in \mathcal{B}_0, a \in G)\}$$

Those (H, r) are called (weakly) degree-preserving functors.

2-morphisms: Let $(H, r), (I, s)\colon \mathcal{B} \to \mathcal{A}$ be degree-preserving functors. Then a natural transformation $\theta\colon H \Longrightarrow I$ is called a *morphism* of degree-preserving functors if $\theta x \in \mathcal{A}^{s_x^{-1} r_x}(Hx, Ix)$ for all $x \in \mathcal{B}$.

Compositions of 1-morphisms: Let $\mathcal{B} \xrightarrow{(H,r)} \mathcal{B}' \xrightarrow{(H',r')} \mathcal{B}''$ be degree-preserving functors. Then

$$(H'H, (r_x r'_{Hx})_{x \in \mathcal{B}})\colon \mathcal{B} \to \mathcal{B}''$$

is also a degree-preserving functor, which we define to be the *composite* $(H', r')(H, r)$ of (H, r) and (H', r').

Vertical and horizontal compositions of 2-morphisms: These are given by the usual ones of natural transformations.

Definition 5. Let $\mathcal{A}$ be a category and $\mathcal{B}$ a G-graded category.

(1) Let $E, F\colon \mathcal{A} \to \mathcal{B}$ be functors. Then a natural transformation $\varepsilon\colon E \Rightarrow F$ is called *homogeneous* if $\varepsilon_x\colon Ex \to Fx$ are homogeneous in $\mathcal{B}$ for all $x \in \mathcal{A}_0$.

(2) Let $\mathcal{S}$ be a subclass of $\mathcal{B}_0$ and $\mathcal{B}'$ a full subcategory of $\mathcal{B}$ with $\mathcal{B}'_0 = \mathcal{S}$. Then $\mathcal{S}$ (or $\mathcal{B}'$) is said to be *homogeneously dense* in $\mathcal{B}$ if for each $x \in \mathcal{B}_0$ there exists an $x' \in \mathcal{S}$ such that there exists a homogeneous isomorphism $x \to x'$.

(3) A functor $F\colon \mathcal{A} \to \mathcal{B}$ is said to be *homogeneously dense* if the object class $F(\mathcal{A}_0)$ is homogeneously dense in $\mathcal{B}$.

Example 6. Let $\mathcal{B}$ be a G-graded category. If $\mathcal{B}(x, x)$ are local $\Bbbk$-algebras for all $x \in \mathcal{B}_0$, then any dense full subcategory $\mathcal{B}'$ of $\mathcal{B}$ is homogeneously dense.

Theorem 7. *Let $(H, r)\colon \mathcal{B} \to \mathcal{A}$ be a degree-preserving functor in G-**GrCat**. Then the following are equivalent.*

(1) *(H, r) is an equivalence in G-**GrCat**.*

(2) *$H\colon \mathcal{B} \to \mathcal{A}$ is a category equivalence with a quasi-inverse I as a left adjoint both of whose counit $\varepsilon\colon IH \Rightarrow \mathbb{1}_{\mathcal{A}}$ and unit $\eta\colon \mathbb{1}_{\mathcal{B}} \Rightarrow HI$ are homogeneous natural isomorphisms.*

(3) *H is fully faithful and homogeneously dense.*

Proposition 8. *Let $\mathcal{B}, \mathcal{A} \in G$-$\mathbf{GrCat}_0$. Then the following are equivalent.*

(1) *$\mathcal{B} \simeq \mathcal{A}$ in G-**GrCat**.*

(2) *There exist homogeneously dense full subcategories $\mathcal{B}'$ and $\mathcal{A}'$ of $\mathcal{B}$ and $\mathcal{A}$, respectively such that $\mathcal{B}' \cong \mathcal{A}'$ in G-**GrCat**.*

3. G-gradable complexes

Definition 9. Let $\mathcal{B}$ be a G-graded category.

(1) A complex $X^{\bullet} = (X^n, d_X^n)_{n\in\mathbb{Z}}$ of $\mathcal{B}$-modules is called *G-graded* if for each $n \in \mathbb{Z}$, $X^n = (X^n, W_{X^n})$ is a G-graded $\mathcal{B}$-module with a decomposition $W_{X^n} : X^n = \bigoplus_{a\in G} X_a^n$ as $\Bbbk$-modules, and $d_X^n : X^n \to X^{n+1}$ is G-degree-preserving, i.e., $d_X^n(X_a^n) \subseteq X_a^{n+1}$ for all $a \in G$.
(2) A morphism $f^{\bullet} : X^{\bullet} \to Y^{\bullet}$ in the category $\mathcal{C}^{\mathrm{b}}(\operatorname{prj}\mathcal{B})$ is called *G-degree-preserving* if for each $n \in \mathbb{Z}$ the component map $f^n : X^n \to Y^n$ of $f^{\bullet}$ is G-degree-preserving.
(3) By $\mathcal{C}^{\mathrm{b}}(\operatorname{prj}\mathcal{B})_G$ we denote the category of G-graded complexes and G-degree-preserving morphisms.
(4) We set $\mathcal{K}^{\mathrm{b}}(\operatorname{prj}\mathcal{B})_G$ to be the factor category of $\mathcal{C}^{\mathrm{b}}(\operatorname{prj}\mathcal{B})_G$ by the homotopy relation given by G-degree-preserving morphisms, and by $\mathrm{Fgt} : \mathcal{K}^{\mathrm{b}}(\operatorname{prj}\mathcal{B})_G \to \mathcal{K}^{\mathrm{b}}(\operatorname{prj}\mathcal{B})$ the forgetful functor $(X^n, W_{X^n}, d_X^n)_n \mapsto (X^n, d_X^n)_n$.
(5) A complex $X^{\bullet} \in \mathcal{K}^{\mathrm{b}}(\operatorname{prj}\mathcal{B})$ is called *G-gradable* if there exists a complex $Y^{\bullet} \in \mathcal{K}^{\mathrm{b}}(\operatorname{prj}\mathcal{B})_G$ such that $X^{\bullet} \cong \mathrm{Fgt}(Y^{\bullet})$ in $\mathcal{K}^{\mathrm{b}}(\operatorname{prj}\mathcal{B})$.

We cite the following from [1] (see that paper also for terminologies not defined here).

Theorem 10. *Let $\mathcal{C}$ be a G-category, and $P : \mathcal{C} \to \mathcal{C}/G$ the canonical G-covering functor. Then the pushdwon functor $P_{\bullet} : \mathcal{K}^{\mathrm{b}}(\operatorname{prj}\mathcal{C}) \to \mathcal{K}^{\mathrm{b}}(\operatorname{prj}\mathcal{C}/G)$ factors through* Fgt *with an equivalence $P'_{\bullet}$:*

$$\begin{tikzcd} \mathcal{K}^{\mathrm{b}}(\operatorname{prj}\mathcal{C}) \arrow[rr, "P_{\bullet}"] \arrow[dr, "P'_{\bullet}"'] & & \mathcal{K}^{\mathrm{b}}(\operatorname{prj}\mathcal{C}/G). \\ & \mathcal{K}^{\mathrm{b}}(\operatorname{prj}\mathcal{C}/G)_G \arrow[ur, "\mathrm{Fgt}"'] & \end{tikzcd}$$

Let $\mathcal{B}$ be a G-graded category. Then the canonical G-covering $Q : \mathcal{B}\#G \to \mathcal{B}$ is the composite of the canonical G-covering $\mathcal{B}\#G \to (\mathcal{B}\#G)/G$ and a degree-preserving equivalence $\omega'_{\mathcal{B}} = (\omega'_{\mathcal{B}}, r_{\mathcal{B}}) : (\mathcal{B}\#G)/G \to \mathcal{B}$, which is an equivalence in G-**GrCat**. Using this fact we have the following.

Corollary 11. *Let $\mathcal{B}$ be a G-graded category, and $Q : \mathcal{B}\#G \to \mathcal{B}$ the canonical G-covering functor. Then the pushdwon functor $Q_{\bullet} : \mathcal{K}^{\mathrm{b}}(\operatorname{prj}\mathcal{B}\#G) \to \mathcal{K}^{\mathrm{b}}(\operatorname{prj}\mathcal{B})$ factors through* Fgt *with an equivalence $Q'_{\bullet}$:*

$$\begin{tikzcd} \mathcal{K}^{\mathrm{b}}(\operatorname{prj}\mathcal{B}\#G) \arrow[rr, "P_{\bullet}"] \arrow[dr, "P'_{\bullet}"'] & & \mathcal{K}^{\mathrm{b}}(\operatorname{prj}\mathcal{B}). \\ & \mathcal{K}^{\mathrm{b}}(\operatorname{prj}\mathcal{B})_G \arrow[ur, "\mathrm{Fgt}"'] & \end{tikzcd}$$

The following is immediate from the above.

Corollary 12. *In the setting above let $\mathcal{P}$ be a set of G-gradable complexes in $\mathcal{K}^{\mathrm{b}}(\operatorname{prj}\mathcal{B})$. Then the pushdown $Q_{\bullet}$ restricts to a G-covering functor $Q_{\bullet}|_{\mathcal{U}} : \mathcal{U} \to \mathcal{P}$, where $\mathcal{U}$ is the full subcategory of $\mathcal{K}^{\mathrm{b}}(\operatorname{prj}\mathcal{B}\#G)$ consisting of those $U^{\bullet} \in \mathcal{K}^{\mathrm{b}}(\operatorname{prj}\mathcal{B}\#G)$ such that $P_{\bullet}(U^{\bullet}) \cong X^{\bullet}$ for some $X^{\bullet} \in \mathcal{P}$.*

Remark 13. In the above, for each $X^{\bullet} \in \mathcal{P}$ fix a $U^{\bullet}(X^{\bullet}) \in \mathcal{U}$ with $P_{\bullet}(U^{\bullet}(X^{\bullet})) \cong X^{\bullet}$, then the covering functor $Q_{\bullet}|_{\mathcal{U}}$ above defines a G-grading on $\mathcal{P}$ as follows. For each $X^{\bullet}, Y^{\bullet} \in \mathcal{P}$,

$Q_{\bullet}$ yields the canonical isomorphism

$$Q_{\bullet}^{(1)} \colon \bigoplus_{a \in G} \mathcal{U}(aU^{\bullet}(X^{\bullet}), U^{\bullet}(Y^{\bullet})) \to \mathcal{P}(X^{\bullet}, Y^{\bullet}),$$

which gives us a G-grading $\mathcal{P}(X^{\bullet}, Y^{\bullet}) = \bigoplus_{a \in G} \mathcal{P}^a(X^{\bullet}, Y^{\bullet})$ on $\mathcal{P}$ by setting $\mathcal{P}^a(X^{\bullet}, Y^{\bullet}) := Q_{\bullet}^{(1)}(\mathcal{U}(aU^{\bullet}(X^{\bullet}), U^{\bullet}(Y^{\bullet})))$.

4. Results

We have the following results.

Theorem 14. *Let $(\mathcal{B}, W), (\mathcal{B}', W')$ be G-graded categories. Consider the following conditions.*

(1) *There exists a tilting subcategory $\mathcal{P}$ of $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}(\mathcal{B}, W))$ consisting of G-gradable complexes with a G-grading V defined as in Remark 13 such that the inclusion $(\mathcal{P}, V) \hookrightarrow \mathcal{K}^{\mathrm{b}}(\mathrm{prj}(\mathcal{B}, W))$ is extended to an equivariance in G-**GrCat**, and $(\mathcal{B}', W')$ and $(\mathcal{P}, V)$ are equivariant in G-**GrCat**.*

(2) *The smash products $\mathcal{B}\#G$ and $\mathcal{B}'\#G$ are derived equivalent.*

Then the condition (1) *implies* (2).

A G-covering $(F, \psi) \colon \mathcal{C} \to \mathcal{B}$ from a G-category $\mathcal{C}$ to a G-graded category $\mathcal{B}$ is said to induce a degree-preserving functor with a map $r \colon \mathcal{C}_0 \to G$ if (F, ψ) is the composite of the canonical G-covering $(P, \phi) \colon \mathcal{C} \to \mathcal{C}/G$ and some equivalence $H \colon \mathcal{C}/G \to \mathcal{B}$ such that $(H, r) \colon \mathcal{C}/G \to \mathcal{B}$ is degree-preserving.

Theorem 15. *Let $\mathcal{C} = (\mathcal{C}, X), \mathcal{C}' = (\mathcal{C}', X')$ be G-categories, $\mathcal{B} = (\mathcal{B}, W), \mathcal{B}' = (\mathcal{B}', W')$ G-graded categories and $(F, \psi) \colon \mathcal{C} \to \mathcal{B}, (F', \psi') \colon \mathcal{C}' \to \mathcal{B}'$ G-coverings inducing degree-preserving functors. Then the following are equivalent:*

(1) *There exists a tilting subcategory $\mathcal{U}$ of $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\,\mathcal{C})$ and a G-action Y on $\mathcal{U}$ such that the inclusion $(\mathcal{U}, Y) \hookrightarrow \mathcal{K}^{\mathrm{b}}(\mathrm{prj}(\mathcal{C}, X))$ is extended to a G-equivariant functor, and $(\mathcal{C}', X)$ and $(\mathcal{U}, Y)$ are G-equivariantly equivalent.*

(2) *There exists a tilting subcategory $\mathcal{P}$ of $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\,\mathcal{B})$ consisting of G-gradable complexes and a G-grading V of $\mathcal{P}$ induced by (F, ψ) such that the inclusion $(\mathcal{P}, V) \hookrightarrow \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\,\mathcal{B})$ is extended to a degree-preserving functor, and $(\mathcal{B}', W')$ and $(\mathcal{P}, V)$ are equivalent in G-**GrCat**.*

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