ON REPRESENTATION-FINITE GENDO-SYMMETRIC BISERIAL ALGEBRAS

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Abstract. An algebra is gendo-symmetric if it is isomorphic to the endomorphism ring of a generator over a symmetric algebra. We classify representation-finite gendo-symmetric algebras which are also biserial, and give some elementary properties.

1. Introduction

This is a report on some results presented in [1]. While our original motivation in *loc. cit.* is rather different, for the sake of brevity, our goal here is to give a classification of representation-finite gendo-symmetric biserial algebras. This gives a very computable (due to its biseriality) class of gendo-symmetric algebras.

Throughout, all algebras are finite dimensional over an algebraically closed field K. All modules are finitely generated right modules unless otherwise specified. Paths of quivers are composed from left to right, which is opposite to the direction of composition of maps. Let us start by recall the definitions of various classes of algebras.

Definition 1. [2, 3] An algebra Γ is *gendo-symmetric* if $\Gamma \cong \mathrm{End}_\Lambda(M)$ where M is a generator of the module category $\mathrm{mod}\Lambda$ over a symmetric algebra Λ.

The two classes of typical examples of gendo-symmetric are Auslander algebra of representation-finite symmetric algebras, and the family of Schur algebras $S(n, r)$ with $n \geq r$. The reason for the second class being gendo-symmetric is that it is isomorphic to the endomorphism ring of direct sum of permutation modules (depending on n and r) over the group algebra $K\mathfrak{S}_r$ of symmetric group, and one of these permutation modules is given by $\mathrm{Ind}_{\{1\}}^{K\mathfrak{S}_r} K = K\mathfrak{S}_r$ when $n \geq r$.

Definition 2. A module is *uniserial* if it is left serial and right serial. An algebra is said to be *biserial* if for any indecomposable projective module, its radical is the sum of two uniserial modules whose intersection is simple or zero.

In order to define the next class of algebras, let us introduce the Brauer tree combinatorics first.

Definition 3. A *Brauer tree* is a datum $(G = (V, E), \sigma := (\sigma_v)_{v\in V}, m := (m_v)_{v\in V})$ where

- $G = (V, E)$ is a (finite) graph which is also tree, where V is the set of vertices and E is the set of edges;
- for each $v \in V$, σ_v is a cyclic ordering (permutation) of all the edges incident to v;
- $(m_v)_{v\in V}$ is a series of positive integers so that $m_v = 1$ for all but at most one vertex; each m_v is called the multiplicity of v.

The detailed version of this paper will be submitted for publication elsewhere.

The *exceptional vertex* of (G, σ, m) is the vertex whose multiplicity, which will be called *exceptional multiplicity*, is not equal to 1. In the case when $m_v = 1$ for all $v \in V$ (which will be denoted by $m \equiv 1$), any vertex and its associated multiplicity can be regarded as being exceptional. A vertex of valency 1 will be called a *leaf vertex*, and its attached edge is called a *leaf*.

We will usually just specify a datum by a tuple $(\underline{G}, m)$, where $\underline{G}$ is the graph G equipped with the cyclic orderings $(\sigma_v)_{v\in V}$. We call $\underline{G}$ a *planar tree* for short. The edge immediately before (resp. after) x in the cyclic ordering around v is called the *predecessor* (resp. *successor*) of x around v. Whenever we visualise a planar tree in a picture, we will present the edges emanating from each vertex according to the associated cyclic ordering in the counter-clockwise direction.

Define a set $H := H_{\underline{G}}$ of symbols $(x|y)$ with x, y being edges of G so that they are both incident to a vertex v and y is the successor of x around v. We also call the vertex v in this instance as the *vertex associated to* $(x|y)$ (or *associating vertex of* $(x|y)$). To avoid ambiguity, we take $H_{\underline{G}} := \{(x|x), (\overline{x}|\overline{x})\}$ in the case when $\underline{G}$ has only one edge x.

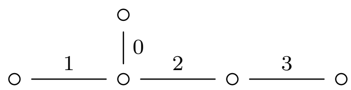
The *Brauer quiver* associated to $\underline{G}$ is a quiver, denoted by $Q_{\underline{G}}$, whose set of vertices is the set E of edges in G, and the set of arrows are given by $x \to y$ for each $(x|y) \in H$. While it makes sense to identify $(Q_{\underline{G}})_1$ with H, there will come a time when we need to distinguish arrows with elements of H. Therefore, we will always denote an arrow by $(x \to y)$ instead of just simply $(x|y)$.

Let $\rho_{x,v} = (x = x_0 \to x_1 \to \cdots \to x_k = x)$ denote the simple cycle (i.e. the one without repeating arrow) in $Q_{\underline{G}}$ so that x_i is incident to v for all $i \in \{1, 2, \ldots, k\}$. Note that k is the valency of the vertex v in G.

Definition 4. An algebra B is a *Brauer tree algebra* associated to the Brauer tree $(\underline{G}, m)$ if it is Morita equivalent to the bounded path algebra $\Lambda_{\underline{G},m} := KQ_{\underline{G}}/I$, where the ideal I is generated by the following Brauer relations:

- $(x \to y \to z) = 0$, if x and z are incident to different vertices;
- $\rho_{x,u}^{m_u} = \rho_{x,v}^{m_v}$, where u, v are the two endpoints of x.

Example 5. Consider the Brauer tree $(\underline{G}, m)$ where $\underline{G}$ has the following visualisation:



This means that the cyclic ordering of the 3-valent vertex u is $(0, 1, 2)$ and the cyclic ordering of the other endpoint v of edge 2 is $(2, 3)$. Suppose the exceptional vertex is v with multiplicity 2. Then all the indecomposable projective modules of $B := \Lambda_{\underline{G},m}$ are uniserial apart from e_2B. The Loewy filtration is described by the cyclic ordering and the multiplicity of the associated vertex, which gives the following pictorial presentation of the Loewy structure of B:

$$\begin{array}{ccccccccc} 0 & & 1 & & & 2 & & & 3 \\ 1 & & 2 & & 0 & & 3 & & 2 \\ 2 & \oplus & 0 & \oplus & 1 & & 2 & \oplus & 3 \\ 0 & & 1 & & & & 3 & & 2 \\ & & & & & 2 & & & 3 \end{array}$$

This presentation gives a clear view of the structures of the two uniserial modules appearing defining the biseriality of B: one has simple subquotients (S_0, S_1, S_2) in its Loewy filtration, while the other has simple subquotients (S_3, S_2, S_3, S_2) in its Loewy filtration.

The following theorem is well-known to experts by combining various results from the literature.

Theorem 6. *An algebra is representation-finite biserial symmetric if, and only if, it is a Brauer tree algebra.*

It turns out that if we replace symmetric by gendo-symmetric on the left-hand side of the theorem, then the right-hand can be replaced by a similarly combinatorial-defined algebra, which we call *special gendo-Brauer tree algebra.*

2. Special gendo-Brauer tree algebras

The following class of modules of Brauer tree algebras plays a central role in our investigation.

Definition 7. Let $B := \Lambda_{\underline{G},m}$ be a basic Brauer tree algebra. For each $(x|y) \in H(\underline{G})$ where $\underline{G}$ has more than one edge, we denote by $M(x,y)$ the hook module $(x \to y)B$. In the case when $\underline{G}$ has only a single edge x (i.e. $B \cong K[X]/(X^{\ell+1})$ for some $\ell \geq 1$), the symbol $M(x,x)$ means either the radical or the socle of the regular module B, i.e. the $K[X]/(X^{\ell+1})$-module isomorphic to either $K[X]/(X^\ell)$ or K. If we ever need to use both of these modules at the same time, we take $\{M(x,x), M(\overline{x},\overline{x})\} = \{K[X]/(X^\ell), K\}$.

The *distance* between two vertices u, v of a planar tree $\underline{G}$ is the number of edges in the (unique) path in G whose endpoints are u, v. Distance gives a natural bipartite structure on trees. In particular, we say that two vertices have the same (resp. different) *parity* if and only if their distance is even (resp. odd).

Definition 8. Fix a Brauer tree $(\underline{G}, m)$ and $H := H_{\underline{G}}$. A (possibly empty) subset W of H is called *special* if there does not exist $(x|y), (y|z) \in W$ so that their associating vertices have different parity.

Fix a special subset W of H. Define a Brauer tree $(\underline{G}^W, m^W) = (G^W, \sigma^W, m^W)$ by enlarging $(\underline{G}, m)$ as follows.

All of the new vertices of $\underline{G}^W$ will have valency 1 and multiplicity 1. The new edges of $\underline{G}^W$ will all be leaves corresponding to elements of W; the endpoints of the (new) leaf corresponding to $(x|y) \in W$ is a new vertex and the vertex associated to $(x|y)$. The cyclic ordering on $\underline{G}^W$ is given by inserting $(x|y)$ in between x and y.

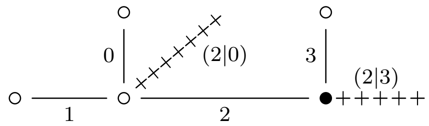
We can visualise the data $(\underline{G}, W)$ in a similar way as Brauer trees by showing the planar tree $\underline{G}^W$ with the following modification:

- Edges in $\underline{G}$ are shown in solid lines with $\circ$ at the endpoints.
- Edges in W (i.e. in $\underline{G}^W \setminus \underline{G}$) are shown in solid lines with "propagation", whereas its attaching leaf vertex will *not* be shown.

Definition 9. For a Brauer tree $(\underline{G}, m)$ with special subset W of $H_{\underline{G}}$, let $\Gamma^W_{\underline{G},m}$ be basic algebra appearing as the quotient of $\Lambda_{\underline{G}^W, m^W}$ by the sum of socles of the projective

$\Lambda_{\underline{G}^W, m^W}$-modules corresponding to $(x|y) \in W \subset (Q_{\underline{G}^W})_0$, over all elements of W. An algebra is called *special gendo-Brauer tree* if it is Morita equivalent to some $\Gamma^W_{\underline{G},m}$.

Example 10. Consider the Brauer tree $(\underline{G}, m)$ given in Example 5. The combinatorial data $(\underline{G}, m, W)$ with $W = \{(2|0), (2|3)\}$ is visualised as follows:



Note that, following the traditional convention in Brauer trees, the exceptional vertex is represented by the black node, in contrast to the other white nodes (vertex with multiplicity 1). It is clear from this visualisation that W is special but not of pure parity. The Loewy structure of the algebra $\Gamma^W_{\underline{G},m}$ is:

$$\begin{array}{cccccccccccc}
 & & & & & & & 2 & & (2|3) & & 3 \\
(2|0) & & 0 & & 1 & & & (2|3) & & 3 & & 2 \\
0 & & 1 & & 2 & & (2|0) & 3 & & 2 & & (2|3) \\
1 & \oplus & 2 & \oplus & (2|0) & \oplus & 0 & 2 & \oplus & (2|3) & \oplus & 3 \\
2 & & (2|0) & & 0 & & 1 & (2|3) & & 3 & & 2 \\
 & & 0 & & 1 & & & 3 & & 2 & & (2|3) \\
 & & & & & & & 2 & & & & 3
\end{array}$$

Let e be the idempotent given by $e_0+e_1+e_2+e_3$. Roughly speaking, the effect of applying the Schur functor $(-)e$ to $\Gamma^W_{\underline{G},m}$ is to remove all the composition factors labelled by $(2|0)$ and $(2|3)$. The resulting diagram is then the same as the Loewy diagram of the generator $\Lambda_{\underline{G},m} \oplus M(2,0) \oplus M(2,3)$ over $\Lambda_{\underline{G},m}$ - this is, in fact, the Morita-Tachikawa correspondence (see [2, 4]) in action.

Theorem 11. *Let $\Gamma^W_{\underline{G},m}$ be a basic special gendo-Brauer tree algebras as in the previous definition. Then $\Gamma^W_{\underline{G},m}$ is isomorphic to the endomorphism ring of the generator*

$$\Lambda_{\underline{G},m} \oplus \bigoplus_{(x|y)\in W} M(x,y)$$

of $\operatorname{mod} \Lambda_{\underline{G},m}$. *In particular, it is representation-finite gendo-symmetric biserial. Conversely, any representation-finite gendo-symmetric biserial algebra appears in such a form up to Morita equivalence.*

We finish by giving some facts about various homological dimensions of these algebras.

Proposition 12. *Let $(\underline{G}, m)$ be a Brauer tree with special subset W. Then the following statements hold.*

(1) $\Gamma^W_{\underline{G},m}$ *has dominant dimension at least 2.*
(2) $\Gamma^W_{\underline{G},m}$ *is Gorenstein, i.e. the injective dimensions of the regular left module and the regular right module are both finite.*
(3) $\Gamma^W_{\underline{G},m}$ *has finite global dimension if, and only if, $m \equiv 1$ and $|W| = 1$.*

In fact, these homological dimensions can be determined combinatorially using $(\underline{G}, m)$ and W. We will omit these description due to technicalities but refer the reader to [1].

References

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