ON THE RELATIONS FOR GROTHENDIECK GROUPS OF COHEN-MACAULAY MODULES OVER GORENSTEIN RINGS

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Abstract. We consider the converse of the Butler, Auslander-Reiten's Theorem which is on the relations for Grothendieck groups. We show that a Gorenstein ring is of finite representation type if the Auslander-Reiten sequences generate the relations for Grothendieck groups. This gives an affirmative answer of the conjecture due to Auslander.

1. Introduction

Throughout this section, $(R, \mathfrak{m})$ denote a commutative Cohen-Macaulay complete ring with the residue field k. All R-modules are assumed to be finitely generated. We say that an R-module M is Cohen-Macaulay if

$$\mathrm{Ext}_R^i(k, M) = 0 \quad \text{for all } i < \dim R.$$

We denote by $\mathrm{mod}(R)$ the category of R-modules with R-homomorphisms and by $\mathcal{C}$ the full subcategory of $\mathrm{mod}(R)$ consisting of all Cohen-Macaulay R-modules.

Set $\mathrm{G}(\mathcal{C}) = \bigoplus_{X \in \mathrm{ind}\mathcal{C}} \mathbb{Z} \cdot [X]$, which is a free abelian group generated by isomorphism classes of indecomposable objects in $\mathcal{C}$. We denote by $\mathrm{EX}(\mathcal{C})$ a subgroup of $\mathrm{G}(\mathcal{C})$ generated by

$$\{[X] + [Z] - [Y] \mid \text{there is an exact sequence } 0 \to Z \to Y \to X \to 0 \text{ in } \mathcal{C}\}.$$

We also denote by $\mathrm{AR}(\mathcal{C})$ a subgroup of $\mathrm{G}(\mathcal{C})$ generated by

$$\{[X] + [Z] - [Y] \mid \text{there is an AR sequence } 0 \to Z \to Y \to X \to 0 \text{ in } \mathcal{C}\}.$$

Let $K_0(\mathcal{C})$ be a Grothendieck group of $\mathcal{C}$. By the definition, $K_0(\mathcal{C}) = \mathrm{G}(\mathcal{C})/\mathrm{EX}(\mathcal{C})$.

On the relation for Grothendieck groups, Butler[3], Auslander-Reiten[2], and Yoshino[7] prove the following theorem.

Theorem 1. [3, 2, 7] *If R is of finite representation type then* $\mathrm{EX}(\mathcal{C}) = \mathrm{AR}(\mathcal{C})$.

Here we say that R is of finite representation type if there are only a finite number of isomorphism classes of indecomposable Cohen-Macaulay R-modules.

Auslander conjectured the converse of Theorem 1 is true. Actually it has been proved by Auslander[1] for Artin algebras and by Auslander-Reiten[2] for complete one dimensional domain. In this note we consider for the case of complete Gorenstein local rings with an isolated singularity. Actually we shall show the following theorem.

Theorem 2. [5] *Let R be a complete Gorenstein local ring with an isolated singularity and with algebraically closed residue field. If* $\mathrm{EX}(\mathcal{C}) = \mathrm{AR}(\mathcal{C})$, *then R is of finite representation type.*

2. Proof of Theorem 2

In the rest of the note, we always assume that $(R, \mathfrak{m})$ is a complete Gorenstein local ring with the residue field k. For the category of Cohen-Macaulay R-modules $\mathcal{C}$, we denote by $\underline{\mathcal{C}}$ the stable category of $\mathcal{C}$. The objects of $\underline{\mathcal{C}}$ are the same as those of $\mathcal{C}$, and the morphisms of $\underline{\mathcal{C}}$ are elements of $\underline{\mathrm{Hom}}_R(M, N) = \mathrm{Hom}_R(M, N)/P(M, N)$ for $M, N \in \underline{\mathcal{C}}$, where $P(M, N)$ denote the set of morphisms from M to N factoring through free R-modules. Since R is complete, $\mathcal{C}$, hence $\underline{\mathcal{C}}$, is a Krull-Schmidt category. For $M \in \mathcal{C}$ we denote it by $\underline{M}$ to indicate that it is an object of $\underline{\mathcal{C}}$. For a finitely generated R-module M, take a free presentation

$$\cdots \to F_1 \xrightarrow{d} F_0 \to M \to 0.$$

We denote Im d by ΩM, which is called a (first) syzygy of M. And we also denote by $\mathrm{tr}M$ the cokernel $F_0^* \xrightarrow{d^*} F_1^*$ where $(-)^* = \mathrm{Hom}_R(-, R)$.

Lemma 3. [5, Lemma 2.1] *There exists $X \in \mathcal{C}$ such that $\underline{\mathrm{Hom}}_R(M, X) \neq 0$ for all $M \in \mathcal{C}$ with $\underline{M} \neq \underline{0}$ in $\underline{\mathcal{C}}$.*

Proof. Take a Cohen-Macaulay approximation of the residue field k as X. One can show that X satisfies the assumption of the lemma. □

The stable category $\underline{\mathcal{C}}$ has a structure of a triangulated category since R is Gorenstein (cf. [4]). By the definition of a triangle, $\underline{L} \to \underline{M} \to \underline{N} \to \underline{L}[1]$ is a triangle in $\underline{\mathcal{C}}$ if and only if there is an exact sequence $0 \to L \to M' \to N \to 0$ in $\mathcal{C}$ with $\underline{M'} \cong \underline{M}$ in $\underline{\mathcal{C}}$. To prove our theorem, we use a theory of Auslander-Reiten (abbr. AR) triangles. The notion of AR triangles is a stable analogy of AR sequences.

Definition 4. [4, Chapter I, §4] We say that a triangle $\underline{Z} \to \underline{Y} \xrightarrow{\underline{f}} \underline{X} \xrightarrow{\underline{w}} \underline{Z}[1]$ in $\underline{\mathcal{C}}$ is an AR triangle ending in $\underline{X}$ (or starting from $\underline{Z}$) if it satisfies

(1) $\underline{X}$ and $\underline{Z}$ are indecomposable.
(2) $\underline{w} \neq 0$.
(3) If $\underline{g} : \underline{W} \to \underline{X}$ is not a split epimorphism, then there exists $\underline{h} : \underline{W} \to \underline{Y}$ such that $\underline{g} = \underline{f} \circ \underline{h}$.

Remark 5. Let $0 \to Z \to Y \xrightarrow{f} X \to 0$ be an AR sequence in $\mathcal{C}$. Then $\underline{Z} \to \underline{Y} \xrightarrow{\underline{f}} \underline{X} \to \underline{Z}[1]$ is an AR triangle in $\underline{\mathcal{C}}$. See [6, Proposition 2.2] for example.

We say that $(R, \mathfrak{m})$ is an isolated singularity if each localization $R_\mathfrak{p}$ is regular for each prime ideal $\mathfrak{p}$ with $\mathfrak{p} \neq \mathfrak{m}$. Note that if R is an isolated singularity, $\mathcal{C}$ admits AR sequences (cf. [7, Theorem 3.2]). Hence $\underline{\mathcal{C}}$ admits AR triangles (Remark 5). We also note that since we have the isomorphism $\underline{\mathrm{Hom}}_R(M, N) \cong \mathrm{Tor}_1^R(\mathrm{tr}M, N)$ for finitely generated R-modules M and N, one can show that $\mathrm{length}_R(\underline{\mathrm{Hom}}_R(M, N))$ is finite for M, $N \in \underline{\mathcal{C}}$ if R is an isolated singularity. When U is indecomposable in $\mathcal{C}$ then denote by $\mu(\underline{U}, \underline{X})$ the multiplicity of $\underline{U}$ as a direct summand of $\underline{X}$. We also denote by $[\underline{U}, \underline{X}]$ the integer $\mathrm{length}_R(\underline{\mathrm{Hom}}_R(U, X))$.

Proposition 6. [5, Proposition 2.4][6, Proposition 2.14 (1)] *Let R be an isolated singularity and let $\underline{Z} \xrightarrow{g} \underline{Y} \xrightarrow{f} \underline{X} \to \underline{Z}[1]$ be an AR triangle in $\underline{\mathcal{C}}$. Then the following equality holds for each indecomposable $U \in \mathcal{C}$:*

$$[\underline{U}, \underline{X}] + [\underline{U}, \underline{Z}] - [\underline{U}, \underline{Y}] = \mu(\underline{U}, \underline{X}) \cdot \dim_k k_{\underline{X}} + \mu(\underline{U}, \underline{\Omega X}) \cdot \dim_k k_{\underline{\Omega X}},$$

where $k_{\underline{X}} = \underline{\mathrm{End}}_R(X)/\mathrm{rad}\underline{\mathrm{End}}_R(X)$.

Proof of Theorem 2. Let X be the module that satisfies the conditions as in Lemma 3. Take the syzygy of X.

$$0 \to \Omega X \to P \to X \to 0.$$

By the assumption, since $\mathrm{EX}(\mathcal{C}) = \mathrm{AR}(\mathcal{C})$, we have the equality in $\mathrm{G}(\mathcal{C})$,

$$[X] + [\Omega X] - [P] = \sum_{M \in \mathrm{ind}\mathcal{C}}^{finite} a_{M,X}([M] + [\tau M] - [E_M]),$$

where $[M] + [\tau M] - [E_M]$ come from AR sequences $0 \to \tau M \to E_M \to M \to 0$. The equality yields that

$$[\underline{U}, \underline{X} \oplus \underline{\Omega X}] = \sum_{M \in \mathrm{ind}\mathcal{C}}^{finite} a_{M,X}([\underline{U}, \underline{M}] + [\underline{U}, \underline{\tau M}] - [\underline{U}, \underline{E_M}]) \tag{2.1}$$

for each $U \in \mathcal{C}$. Since $\underline{\tau M} \to \underline{E_M} \to \underline{M} \to \underline{\tau M}[1]$ are AR triangles (Remark 5), by Proposition 6, we see that there are only a finite number of indecomposable modules in $\mathcal{C}$ that makes the RHS in (2.1) non-zero, so is LHS. By Lemma 3, we conclude that $\underline{\mathcal{C}}$, hence $\mathcal{C}$ is of finite representation type. $\square$

References

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