PERSISTENT HOMOLOGY AND AUSLANDER-REITEN THEORY

YASUAKI HIRAOKA

ABSTRACT. The content of this report is based on a series of papers [4, 5] which present an extension of persistent homology as representations on quivers with nontrivial relations. We aim to briefly explain the connection between Auslander-Reiten theory and persistent homology.

1. INTRODUCTION

Let X_r and Y_r be two topological spaces parametrized by $r \in \mathbb{N} = \{1, 2, \dots\}$. Suppose that, for parameters $r < s$, we have continuous maps $f : X_r \to X_s$ and $g : Y_r \to Y_s$. By taking homology $f_* : H_*(X_r) \to H_*(X_s)$ and $g_* : H_*(Y_r) \to H_*(Y_s)$ with field coefficient K, we can study persistent topological features in the parameter interval $[r, s]$ for both X and Y. This is the simplest setting of persistent homology [3, 10], and is characterized by a unique decomposition

$$(f_* : H_*(X_r) \to H_*(X_s)) \;\simeq\; I(r,r)^{n_r} \oplus I(r,s)^{n_{rs}} \oplus I(s,s)^{n_s},$$

where $n_r, n_{rs}, n_s \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, and $I(r,r), I(r,s)$, and $I(s,s)$ are indecomposable modules defined by

$$I(r,r) : K \to 0,\ I(r,s) : K \to K,\ I(s,s) : 0 \to K.$$

Similarly, we obtain a decomposition for $g_* : H_*(Y_r) \to H_*(Y_s)$. These decompositions characterize "persistence" in the sense that topological features specified by the summand $I(r,s)$ are regarded as robust in the parameter interval $[r, s]$, and those specified by the summands $I(r,r)$ and $I(s,s)$ are observed only at each parameter value. Our interest in this paper is to compare the persistent topological features of $f : X_r \to X_s$ and $g : Y_r \to Y_s$.

For this purpose, it is natural to map X_r and Y_r into another topological space Z_r as

$$X_r \to Z_r \leftarrow Y_r,$$

and study common topological features via Z_r. One of the standard choices for these maps is the inclusion $X_r \hookrightarrow X_r \cup Y_r \hookleftarrow Y_r$. Then, in order to study robust and common topological features, a commutative diagram of topological spaces

$$\begin{array}{ccccc} X_s & \longrightarrow & Z_s & \longleftarrow & Y_s \\ \uparrow & & \uparrow & & \uparrow \\ X_r & \longrightarrow & Z_r & \longleftarrow & Y_r \end{array} \qquad (1.1)$$

provides an appropriate geometric setting. It should be noted that the individual robust topological features are measured in the vertical direction and the common topological features are measured in the horizontal direction. Hence, on the homological level, our problem is to consider a commutative diagram

(1.2)
$$\begin{tikzcd} H_*(X_s) \arrow[r] & H_*(Z_s) & H_*(Y_s) \arrow[l] \\ H_*(X_r) \arrow[r] \arrow[u] & H_*(Z_r) \arrow[u] & H_*(Y_r) \arrow[l] \arrow[u] \end{tikzcd}$$

of K-vector spaces and linear maps between them.

The algebraic object (1.2) can be regarded as a representation of the quiver

(1.3)
$$\begin{tikzcd} \circ \arrow[r] & \circ & \circ \arrow[l] \\ \circ \arrow[r] \arrow[u] & \circ \arrow[u] & \circ \arrow[l] \arrow[u] \end{tikzcd}$$

with commutative relations. We call this quiver the commutative triple ladder in this paper. Recall that a representation V of this quiver is given by a commutative diagram

(1.4)
$$\begin{tikzcd} V_4 \arrow[r, "f_{45}"] & V_5 & V_6 \arrow[l, "f_{65}"'] \\ V_1 \arrow[r, "f_{12}"'] \arrow[u, "f_{14}"] & V_2 \arrow[u, "f_{25}"] & V_3 \arrow[l, "f_{32}"] \arrow[u, "f_{36}"] \end{tikzcd}$$

of finite dimensional K-vector spaces and linear maps between them. In view of the correspondence between representations and modules, we call (1.4) a persistence module [1, 2, 4, 10] on the commutative ladder (1.3). Then, our problem is to consider decompositions of a given persistence module.

Under this setting, the paper [4] clarifies the following:

(1) The number of isomorphism classes of indecomposable persistence modules on (1.3) is finite. This implies that persistence modules on (1.3) can be classified by complete discrete invariants.

(2) The Auslander-Reiten quiver of (1.3), which lists up all the isomorphism classes of indecomposable persistence modules and irreducible morphisms among them, is explicitly derived. Moreover, the notion of persistence diagrams is generalized to functions on the vertex set of the Auslander-Reiten quiver. In particular, the multiplicity of the persistence module

$$\begin{tikzcd} K \arrow[r] & K & K \arrow[l] \\ K \arrow[r] \arrow[u] & K \arrow[u] & K, \arrow[l] \arrow[u] \end{tikzcd}$$

where all the maps are identity maps, characterizes the robust and common topological features between $f : X_r \to X_s$ and $g : Y_r \to Y_s$.

(3) An algorithm for computing indecomposable decompositions is presented by using the Auslander-Reiten quiver.

(4) Numerical examples to detect robust and common topological features are shown.

We note that the quiver (1.3) is not one of A_n, D_n and E_6, E_7, E_8 types studied in Gabriel's theorem [6]. However, the commutativity imposed in (1.3) enables us to derive the finiteness property of the representation type as mentioned above.

The research in [4] is motivated both by a recent theoretical result and by applications of topological data analysis. On the theoretical side, it is inspired by the paper [1], where Carlsson and de Silva formulate persistent homology as representations of an A_n quiver and generalize to zigzag persistence. By this connection to the representation theory, the algebraic aspects of persistent homology have been understood further. The paper [4] deals with a generalization to representations of associative algebras.

On the other hand, from the application side, this work is motivated by the analysis of protein [7] and amorphous structures [8, 9] using persistent homology. For example, in the paper [7], the authors study a topological characterization of protein compressibility by using persistent homology computed on α-complex models of proteins. One of the key observations obtained in that work is that there exist two distinct radius parameters $r < s$ for the atoms which characterize protein compressibility. Hence, once we develop a tool which gives us more detailed information of robust and common topological features on these restricted two parameter values, we expect to obtain further understanding of the relationship between protein compressibility and their geometric structures. The result in the paper [4] provides a tool for that purpose.

2. Auslander-Reiten theory on Commutative Triple Ladder

In this section, we first provide the explicit form of the Auslander-Reiten quiver of the commutative ladder (1.3). Then, from the finiteness of the Auslander-Reiten quiver, a generalization of persistence diagrams for the commutative ladder (1.3) is naturally derived.

On the commutative triple ladder, let us denote a persistence module

$$\begin{tikzcd}
K^{n_4} \arrow[r] & K^{n_5} & K^{n_6} \arrow[l] \\
K^{n_1} \arrow[r] \arrow[u] & K^{n_2} \arrow[u] & K^{n_3} \arrow[l] \arrow[u]
\end{tikzcd}$$

by $\begin{smallmatrix} n_4 n_5 n_6 \\ n_1 n_2 n_3 \end{smallmatrix}$. This is called its dimension vector expression. In the paper [4], the following theorem is presented.

Theorem 1. *The Auslander-Reiten quiver* Γ *of the commutative triple ladder* (1.3) *is given as follows:*
(2.1)

$$\begin{tikzcd}[column sep=small]
 & & & & \substack{011\\011} \arrow[dr] & & \substack{100\\000} \arrow[dr] & & \substack{000\\110} \arrow[dr] & & \substack{001\\001} \arrow[dr] & \\
 & \substack{110\\000} \arrow[dr] & & \substack{011\\010} \arrow[ur] \arrow[dr] & & \substack{111\\011} \arrow[ur] \arrow[dr] & & \substack{100\\110} \arrow[ur] \arrow[dr] & & \substack{001\\111} \arrow[ur] \arrow[dr] & & \substack{000\\001} \\
\substack{010\\000} \arrow[ur] \arrow[r] \arrow[dr] & \substack{010\\010} \arrow[r] & \substack{121\\010} \arrow[ur] \arrow[r] \arrow[dr] & \substack{111\\000} \arrow[r] & \substack{111\\010} \arrow[ur] \arrow[r] \arrow[dr] & \substack{000\\010} \arrow[r] & \substack{111\\121} \arrow[ur] \arrow[r] \arrow[dr] & \substack{111\\111} \arrow[r] & \substack{101\\111} \arrow[ur] \arrow[dr] & & \substack{000\\111} \arrow[ur] \arrow[dr] & \\
 & \substack{011\\000} \arrow[ur] & & \substack{110\\010} \arrow[ur] \arrow[dr] & & \substack{111\\110} \arrow[ur] \arrow[dr] & & \substack{001\\011} \arrow[ur] \arrow[dr] & & \substack{100\\111} \arrow[ur] \arrow[dr] & & \substack{000\\100} \\
 & & & & \substack{110\\110} \arrow[ur] & & \substack{001\\000} \arrow[ur] & & \substack{000\\011} \arrow[ur] & & \substack{100\\100} \arrow[ur] &
\end{tikzcd}$$

The morphisms on $K \to K$ *are all identity maps, and the maps in* $\substack{111\\121}$ *and* $\substack{121\\010}$ *are given by*

$$\begin{tikzcd}
K \arrow[r] & K & K \arrow[l] \\
K \arrow[u] \arrow[r, "f_{12}"'] & K^2 \arrow[u, "f_{25}"] & K \arrow[u] \arrow[l, "f_{32}"]
\end{tikzcd}, \quad f_{12} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ f_{32} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ f_{25} = \begin{pmatrix} 1 & 1 \end{pmatrix},$$

$$\begin{tikzcd}
K \arrow[r, "f_{45}"] & K^2 & K \arrow[l, "f_{65}"'] \\
0 \arrow[u] \arrow[r] & K \arrow[u, "f_{25}"] & 0 \arrow[u] \arrow[l]
\end{tikzcd}, \quad f_{45} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ f_{65} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ f_{25} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

From Theorem 1, all the indecomposable modules are classified by the dimension vectors listed in the above, and any persistence module V can be uniquely expressed by

$$V \simeq \bigoplus_{[M]\in\Gamma} [M]^{n_{[M]}}, \tag{2.2}$$

where $[M] \in \Gamma$ is a vertex in Γ and $n_{[M]} \in \mathbb{N}_0$. Hence, these dimension vectors are the complete discrete invariants for persistence modules on (1.3). In particular, the dimension vector $\substack{111\\111}$ gives us information about robust common topological features.

From this remark, the following is a natural generalization of persistence diagrams.

Definition 2. The persistence diagram $D(V)$ of a persistence module (2.2) on the commutative triple ladder is a function on the vertex set of the Auslander-Reiten quiver Γ:

$$D(V) = \{n_{[M]} \in \mathbb{N}_0 \mid [M] : \text{vertex of } \Gamma\}.$$

That is, the value of $D(V)$ at $[M]$ is given by $n_{[M]}$ in the indecomposable decomposition of V in (2.2).

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WPI-Advanced Institute for Materials Research
Tohoku University
2-1-1 Katahira, Aoba-ku, Sendai, 980-8577 Japan
E-mail address: hiraoka@wpi-aimr.tohoku.ac.jp