# 3-DIMENSIONAL QUADRATIC ARTIN-SCHELTER REGULAR ALGEBRAS AND SUPERPOTENTIALS

AYAKO ITABA

Abstract. Let $k$ be an algebraically closed field of characteristic 0, $A$ a graded $k$-algebra finitely generated in degree 1 and $V$ a $k$-vector space. For the 3-dimensional quadratic AS-regular algebra $A$, we consider the following two conjectures: (I) there exist a superpotential $w \in V^{\otimes 3}$ and an automorphism $\tau$ of $V$ such that $A$ and the derivation-quotient algebra $\mathcal{D}(w^\tau)$ of $w^\tau$ are isomorphic as graded algebras; (II) there exists a Calabi-Yau AS-regular algebra $C$ such that $A$ and $C$ are graded Morita equivalent. In this talk, we give partial results for the above two conjectures.

## 1. AS-regular algebras and geometric algebras

Through this report, let $k$ be an algebraically closed field of characteristic 0, $A$ a graded $k$-algebra finitely generated in degree 1. That is, $A = T(V)/I$, where $V$ is a $k$-vector space, $T(V)$ is the tensor algebra of $V$ and $I$ is a two-sided ideal of $T(V)$ with $I_0 = I_1 = 0$.

Artin and Schelter [1] defined certain regular algebras (called *Artin-Schelter regular algebras* later). Moreover, Artin, Tate and Van den Bergh [2] classified Artin-Schelter regular algebras of global dimension 3 by geometry. First, we recall the definition of Artin-Schelter regular algebras as follows:

**Definition 1.** ([1]) Let $A$ be a noetherian connected graded $k$-algebra. $A$ is called a *$d$-dimensional Artin-Schelter regular* (simply *AS-regular*) *algebra* if $A$ satisfies the following conditions:

(i) $\operatorname{gldim} A = d < \infty$,
(ii) $\operatorname{GKdim} A := \inf\{\alpha \in \mathbb{R} \mid \dim_k(\sum_{i=0}^{n} A_i) \leq n^\alpha \text{ for all } n \gg 0\} < \infty$, called the *Gelfand-Kirillov dimension* of $A$,
(iii) (*Gorenstein condition*) $\operatorname{Ext}^i_A(k, A) = \begin{cases} k & (i = d), \\ 0 & (i \neq d). \end{cases}$

For example, let $A$ is a grade $k$-algebra

$$k\langle x, y, z\rangle/(yz - \alpha zy, zx - \beta xz, xy - \gamma yx)\ (\alpha\beta\gamma \neq 0).$$

Then, $A$ is a 3-dimensional AS-regular algebra. For another example, if $A$ is a graded commutative algebra, $A$ is $n$-dimensional AS-regular if and only if $A$ is isomorphic to a polynomial ring $k[x_1, \dots, x_n]$.

Next, we recall the definition of Koszul algebras.

---

The detailed version of this paper will be submitted for publication elsewhere.

**Definition 2.** Let $A$ be a graded $k$-algebra. A graded $A$-module $M$ has a *linear resolution* if a free resolution of $M$ is as follows:

$$\cdots \longrightarrow \bigoplus A(-3) \longrightarrow \bigoplus A(-2) \longrightarrow \bigoplus A(-1) \longrightarrow \bigoplus A \longrightarrow M \longrightarrow 0.$$

A graded $k$-algebra $A$ is called *Koszul* when $k$ has a linear resolution.

*Remark* 3. If $A$ is a Koszul algebra, then $A = T(V)/(R)$ is quadratic, where $R \subset V \otimes_k V$. Moreover, the Ext algebra (the Yoneda algebra) of $A$ $\mathrm{Ext}_A^*(k,k) \cong A^! := T(V^*)/(R^\perp)$ is Koszul, and $A^!$ is called the *Koszul dual* of $A$, where $V^*$ is the dual space of a finite-dimensional $k$-vector space $V$, and $R^\perp := \{f \in V^* \otimes_k V^* \mid f(R) = 0\}$.

For example, let $A$ be a graded $k$-algebra

$$k\langle x, y, z\rangle/(yz - \alpha zy, zx - \beta xz, xy - \gamma yx).$$

Then, the Koszul dual $A^!$ of $A$ is

$$k\langle x, y, z\rangle/(x^2, y^2, z^2, \alpha yz + zy, \beta zx + xz, \gamma xy + yx) \quad (\alpha, \beta, \gamma \in k\backslash\{0\}).$$

For Koszul algebras, by using Koszul duality, Smith [8] proved a relationship between AS-regular Koszul algebras and Frobenius Koszul algebras.

**Theorem 4.** ([8, Proposition 5.10]) *Let $A$ be a connected graded Koszul $k$-algebra. Then $A$ is Koszul AS-regular if and only if the Koszul dual $A^!$ is Frobenius and the complexity of $k$ is finite.*

We remark that, in Theorem 4, for a $d$-dimensional AS-regular Koszul algebra $A$ and the Frobenius Koszul algebra $A^!$, $\mathrm{gldim}\, A \le d$ and $\mathrm{GK\,dim}\, A = l < \infty$ correspond to $(\mathrm{rad}\, A^!)^{d+1} = 0$ and $\mathrm{cx}(A^!/\mathrm{rad}\, A^!) = \mathrm{cx}(k) = l < \infty$, respectively. For example,

$$A = k\langle x, y, z\rangle/(yz - \alpha zy, zx - \beta xz, xy - \gamma yx) \quad (\alpha\beta\gamma \neq 0)$$

is a 3-dimensional AS-regular Koszul algebra. Moreover,

$$A^! = k\langle x, y, z\rangle/(x^2, y^2, z^2, \alpha yz + zy, \beta zx + xz, \gamma xy + yx) \quad (\alpha, \beta, \gamma \in k\backslash\{0\})$$

is a Frobenius Koszul algebra.

Now, we consider a homogeneous ideal $I$ of $k\langle x_1, \dots, x_n\rangle$ generated by degree 2 homogeneous polynomials, that is, we treat a quadratic algebra. When a graded $k$-algebra $A = k\langle x_1, \dots, x_n\rangle/I$ is quadratic, we set

$$\Gamma_A := \{(p, q) \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \mid f(p, q) = 0 \text{ for all } f \in I_2\}.$$

I.Mori [5] introduced a geometric algebra over $k$ as follows.

**Definition 5.** ([5]) Let $A = k\langle x_1, \dots, x_n\rangle/I$ be a quadratic $k$-algebra.

(i) $A$ satisfies (G1) if there exists a pair $(E, \sigma)$ where $E$ is a closed $k$-subscheme of $\mathbb{P}^{n-1}$ and $\sigma \in \mathrm{Aut}\, E$ such that

$$\Gamma_A = \{(p, \sigma(p)) \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \mid p \in E\}.$$

In this case, we write $\mathcal{P}(A) = (E, \sigma)$ called *the geometric pair of $A$*.

(ii) $A$ satisfies (G2) if there exists a pair $(E, \sigma)$ where $E$ is a closed $k$-subscheme of $\mathbb{P}^{n-1}$ and $\sigma \in \operatorname{Aut} E$ such that

$$I_2 = \{f \in k\langle x_1, \dots, x_n\rangle_2 \mid f(p, \sigma(p)) = 0, \text{ for all } p \in E\}.$$

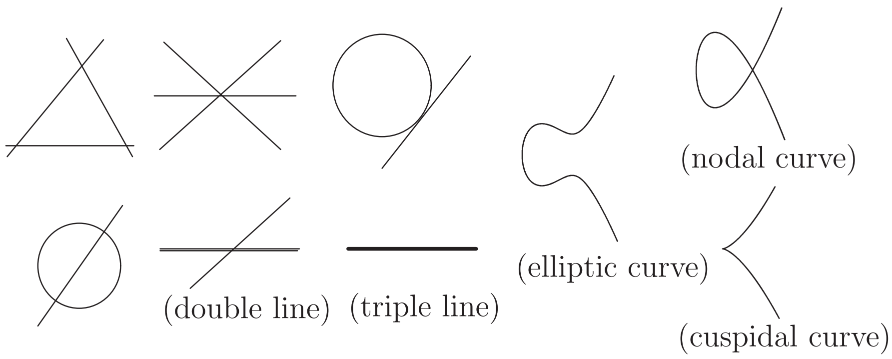
In this case, we write $A = \mathcal{A}(E, \sigma)$.

(iii) $A$ is called *geometric* if $A$ satisfies both (G1) and (G2), and $A = \mathcal{A}(\mathcal{P}(A))$.

Note that, if $A$ satisfies (G1), $A$ determines the pair $(E, \sigma)$ by using $\Gamma_A$. Conversely, if $A$ satisfies (G2), $A$ is determined by the pair $(E, \sigma)$.

In this research, we consider 3-dimensional quadratic AS-regular algebras. These are classified by Artin-Tate-Van den Bergh [2] using a geometric pair $(E, \sigma)$, where $E$ is a cubic curve of $\mathbb{P}^2$ and $\sigma$ is an automorphism of $E$.

**Theorem 6.** ([2]) *Every 3-dimensional quadratic AS-regular algebra $A$ is geometric. Moreover, when $\mathcal{P}(A) = (E, \sigma)$, $E$ is either the projective plane $\mathbb{P}^2$ or a cubic divisor in $\mathbb{P}^2$ as follows.*




## 2. Calabi-Yau algberas and superpotentials

Note that a 3-dimensional quadratic AS-regular algebra is Koszul, and that the quadratic dual $A^!$ of $A$ is a Frobenius algebra by Theorem 4. Then, the Nakayama automorphism of $A^!$ is identity if and only if $A$ is a Calabi-Yau algebra ([7]). Here, the definition of Calabi-Yau algebras is as follows:

**Definition 7.** ([4]) Let $A$ be a connected graded noetherian $k$-algebra $A$ is called *$d$-dimensional Calabi-Yau* if $A$ satisfies the following conditions:

(i) $\operatorname{pd}_{A^e} A = d < \infty$,

(ii) $\operatorname{Ext}^i_{A^e}(A, A^e) = \begin{cases} A & \text{if } i = d, \\ 0 & \text{if } i \neq d, \end{cases}$

where $A^e = A \otimes_k A^{op}$: the enveloping algebra of $A$.

Using a geometric pair $(E, \sigma)$ classified by 6, we determine the algebras $A = \mathcal{A}(E, \sigma)$. Then, we investigate whether these algebras $A$ are graded Morita equivalent to Calabi-Yau AS-regular algebras or not.

Now, we give the definition of a superpotential:

**Definition 8.** ([3], [6]) For a finite-dimensional $k$-vector space $V$, we define the $k$-linear map $\varphi$: $V^{\otimes 3} \longrightarrow V^{\otimes 3}$ by

$$\phi(v_1 \otimes v_2 \otimes v_3) := v_3 \otimes v_1 \otimes v_2.$$

If $\phi(w) = w$ for $w \in V^{\otimes 3}$, then $w$ is called *superpotential.* Also, for $\tau \in \mathrm{GL}(V)$, we define

$$w^{\tau} := (\tau^2 \otimes \tau \otimes \mathrm{id})(w),$$

where $\mathrm{GL}(V)$ is the general linear group of $V$.
Moreover, for a finite-dimensional $k$-vector space $V$ and a subspace $W$ of $V^{\otimes 3}$, we set

- $\partial W := \{(\psi \otimes \mathrm{id}^{\otimes 2})(w) \mid \psi \in V^*,\ w \in W\}$,
- $\mathcal{D}(W) := T(V)/(\partial W)$.

For $w \in V^{\otimes 3}$, $\mathcal{D}(w) := \mathcal{D}(kw)$ is called the *derivation-quotient algebra* of $w$.

## 3. Main results and examples

In this research, our aim is to solve the following two conjectures:

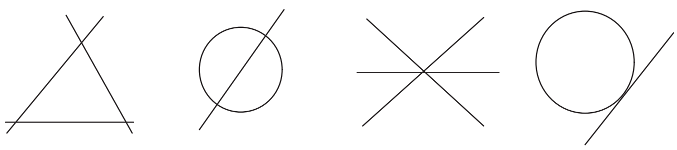
**Conjecture** For every 3-dimensional quadratic AS-regular algebra $A$,

**(I):** there exists a superpotential $w \in V^{\otimes 3}$ and an automorphism $\tau$ of $V$ such that $A$ and the derivation-quotient algebra $\mathcal{D}(w^{\tau})$ of $w^{\tau}$ are isomorphic as graded algebras;

**(II):** there exists a Calabi-Yau AS-regular algebra $C$ such that $A$ and $C$ are graded Morita equivalent.

Our main result is to give partial results for the above two conjectures.

**Theorem 9.** *For the* 3*-dimensional quadratic AS-regular algebra* $A = \mathcal{A}(E, \sigma)$ *corresponding to* $E$ *and* $\sigma \in \mathrm{Aut}\, E$*, suppose that* $E$ *is* $\mathbb{P}^2$ *or the cubic curve of* $\mathbb{P}^2$ *as follows*: *Then, the following* (I) *and* (II) *hold*:



**(I):** *there exist a superpotential* $w \in V^{\otimes 3}$ *and an automorphism* $\tau$ *of* $V$ *such that* $A$ *and the derivation-quotient algebra* $\mathcal{D}(w^{\tau})$ *of* $w^{\tau}$ *are isomorphic as graded algebras;*

**(II):** *there exists a Calabi-Yau AS-regular algebra* $C$ *such that* $A$ *and* $C$ *are graded Morita equivalent.*

**Example 10.** Suppose that $(E, \sigma)$ is a geometric pair where $E$ is a union of three lines making a triangle in $\mathbb{P}^2$ and $\sigma \in \mathrm{Aut}\, E$ stabilizes each component. That is, $E = \mathcal{V}(xyz)$ and

$$\begin{cases} \sigma(\mathcal{V}(x)) = \mathcal{V}(x), \\ \sigma(\mathcal{V}(y)) = \mathcal{V}(y), \\ \sigma(\mathcal{V}(z)) = \mathcal{V}(z). \end{cases}$$

Considering $A = \mathcal{A}(E, \sigma)$ corresponding to $E$ and $\sigma \in \mathrm{Aut}\, E$, $A = k\langle x, y, z\rangle/(yz - \alpha zy, zx - \beta xz, xy - \gamma yx)$ is 3-dimensional quadratic AS-regular ($\alpha, \beta, \gamma \in k\backslash\{0\}$).

For $\lambda := \sqrt[3]{\alpha\beta\gamma} \in k\backslash\{0\}$, we take a superpotential $w$ as

$$w = (xyz + yzx + zxy) - \lambda(zyx + yxz + xzy).$$

Also, we take $\tau := \begin{pmatrix} \sqrt[3]{\beta\gamma^{-1}} & 0 & 0 \\ 0 & \sqrt[3]{\gamma\alpha^{-1}} & 0 \\ 0 & 0 & \sqrt[3]{\alpha\beta^{-1}} \end{pmatrix} \in \mathrm{GL}\,(3,k)$. Then, $w^\tau$ is as follows:

$$\begin{aligned} w^\tau &= (\tau^2 \otimes \tau \otimes \mathrm{id})(w) \\ &= \sqrt[3]{\alpha^{-1}\beta^2\gamma^{-1}}xyz + \sqrt[3]{\alpha^{-1}\beta^{-1}\gamma^2}yzx + \sqrt[3]{\alpha^2\beta^{-1}\gamma^{-1}}zxy \\ &\quad - \sqrt[3]{\alpha^2\beta^{-1}\gamma^2}zyx - \sqrt[3]{\alpha^{-1}\beta^2\gamma^2}yxz - \sqrt[3]{\alpha^2\beta^2\gamma^{-1}}xzy. \end{aligned}$$

Therefore, the derivation-quotient algebra $\mathcal{D}(w^\tau)$

$$\mathcal{D}(w^\tau) = k\langle x, y, z\rangle/(yz - \alpha zy, zx - \beta xz, xy - \gamma yx)$$

and we have a graded $k$-algebra isomorphism $A \cong \mathcal{D}(w^\tau)$ (**Conjecture (I)**).

Moreover, $A$ is equivalent to the Calabi-Yau AS-regular algebra

$$C = k\langle x, y, z\rangle/(yz - \lambda zy, zx - \lambda xz, xy - \lambda yx)$$

as graded Morita equivalent (**Conjecture (II)**).

**Example 11.** Suppose that $(E, \sigma)$ is a geometric pair where $E$ is a union of three lines meeting at one point in $\mathbb{P}^2$ and $\sigma \in \operatorname{Aut} E$ interchange two of its components. That is, $E = \mathcal{V}(xyz)$ and

$$\begin{cases} \sigma(\mathcal{V}(x)) = \mathcal{V}(y), \\ \sigma(\mathcal{V}(y)) = \mathcal{V}(x), \\ \sigma(\mathcal{V}(z)) = \mathcal{V}(z). \end{cases}$$

Considering $A = \mathcal{A}(E, \sigma)$ corresponding to $(E, \sigma)$, $A = k\langle x, y, z\rangle/(f_1, f_2, f_3)$ is 3-dimensional quadratic AS-regular, where $\begin{cases} f_1 &= x^2 - y^2, \\ f_2 &= xz - zy - \beta xy + (\beta - \gamma)y^2, \\ f_3 &= yz - zx - \alpha yx + (\alpha - \gamma)x^2 \end{cases}$ $(\alpha, \beta, \gamma \in k\backslash\{0\})$.

For $\lambda := \frac{1}{3}(\alpha + \beta - \gamma) \in k\backslash\{0\}$, we take a superpotential $w$ as

$$\begin{aligned} w : &= (xyz + yzx + zxy) - (zyx + yxz + xzy) \\ &\quad + \lambda(x^2y + xyx + yx^2) - \lambda(xy^2 + y^2x + yxy). \end{aligned}$$

Also, we take $\tau := \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ \frac{1}{3}(-\alpha + 2\beta - \gamma) & \frac{1}{3}(2\alpha - \beta - \gamma) & -1 \end{pmatrix} \in \mathrm{GL}\,(3,k)$. Therefore, we have a graded $k$-algebra isomorphism $A \cong \mathcal{D}(w^\tau)$ (**Conjecture (I)**).

Moreover, $A$ is equivalent to the Calabi-Yau AS-regular algebra

$$C = k\langle x, y, z\rangle/(g_1, g_2, g_3)$$

as graded Morita equivalent, where

$$\begin{cases} g_1 & = yz - zy + \lambda(xy + yx - y^2), \\ g_2 & = zx - xz + \lambda(x^2 - yx - xy), \quad (\textbf{Conjecture (II)}). \\ g_3 & = xy - yx. \end{cases}$$

## References


[1] M. Artin and W. Schelter, Graded algebras of global dimension 3, Adv. Math. **66** (1987), 171–216.

[2] M. Artin, J. Tate and M. Van den Bergh, Some algebras associated to automorphisms of elliptic curves, The Grothendieck Festschrift, vol. 1, Progress in Mathematics vol. 86 (Brikhäuser, Basel, 1990) 33–85.

[3] R. Bocklandt, T. Schedler and M. Wemyss, Superpotentials and higher order derivations, J. Pure Appl. Algebra **214** (2010), 1501–1522.

[4] V. Ginzburg, Calabi-Yau algebras, arXiv: 0612139 (2007).

[5] I. Mori, Non commutative projective schemes and point schemes, Algebras, Rings and Their Representations, World Sci. Hackensack, N. J., (2006), 215–239.

[6] I. Mori and S. P. Smith, $m$-Koszul Artin-Schelter regular algebras, J. Algebra. **446** (2016), 373–399.

[7] M. Reyes, D. Rogalski and J. J. Zhang, Skew Calabi-Yau algebras and homological identities, Adv. Math. **264** (2014), 308–354.

[8] S. P. Smith, Some finite dimensional algebras related to elliptic curves, in Representation Theory of Algebras and Related Topics (Mexico City, 1994), CMS Conf. Proc, **19**. Amer. Math. Soc., Providence, (1996), 315–348.



Ayako Itaba
Faculty of Science
Shizuoka University
336 Ohya, Shizuoka, 422-8529, Japan
*E-mail address*: itaba.ayako@shizuoka.ac.jp