How to capture t-structures by silting theory

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Aim

Understand bounded t-structures by silting objects.

Silting object

$\mathcal{T}$: triangulated category with shift functor $[1]$
M: object of $\mathcal{T}$

Definition [Keller-Vossieck (1988)]

M: silting object of $\mathcal{T}$:$\Leftrightarrow$

- $\mathrm{Hom}_{\mathcal{T}}(M, M[{}^{\forall}i > 0]) = 0$ ($\Leftrightarrow$: M: presilting)
- $\mathcal{T} = \mathrm{thick}\, M$

$\mathrm{silt}\,\mathcal{T}$: the set of isoclasses of basic silting objects of $\mathcal{T}$

Example

Λ: finite dimensional algebra over a filed
$\Longrightarrow$ Λ is a silting object of $\mathrm{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda)$.

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t-structure

$\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}$: full subcategories of $\mathcal{T}$ closed under isom.
$\mathcal{T}^{\leq n} := \mathcal{T}^{\leq 0}[-n]$, $\mathcal{T}^{\geq n} := \mathcal{T}^{\geq 0}[-n]$ $(n \in \mathbb{Z})$

Definition [Beilinson-Bernstein-Deligne (1982)]

$(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$: t-structure on $\mathcal{T}$ $:\Leftrightarrow$

- $\mathcal{T}^{\leq 0} \subseteq \mathcal{T}^{\leq 1}$ and $\mathcal{T}^{\geq 1} \subseteq \mathcal{T}^{\geq 0}$
- $\mathrm{Hom}_{\mathcal{T}}(X, Y) = 0$ $({}^{\forall}X \in \mathcal{T}^{\leq 0}, {}^{\forall}Y \in \mathcal{T}^{\geq 1})$
- $\mathcal{T} = \mathcal{T}^{\leq 0} * \mathcal{T}^{\geq 1}$

$$:= \left\{ Z \in \mathcal{T} \;\middle|\; \begin{array}{l} \exists \text{tri. } X \to Z \to Y \to X[1] \\ (X \in \mathcal{T}^{\leq 0}, Y \in \mathcal{T}^{\geq 1}) \end{array} \right\}$$

The heart $\mathcal{T}^0 := \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$ is abelian.

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Bounded t-structure

Definition

$(\mathcal{T}^{\le 0}, \mathcal{T}^{\ge 0})$: <u>bounded</u> $:\Leftrightarrow \mathcal{T} = \bigcup_{n\in\mathbb{Z}} \mathcal{T}^{\le n} = \bigcup_{n\in\mathbb{Z}} \mathcal{T}^{\ge n}$

$\Leftrightarrow \mathcal{T} = \mathsf{thick}\,\mathcal{T}^0$

$t\text{-}\mathrm{str}_{\mathrm{bd}}\mathcal{T}$: the set of bounded t-structures on $\mathcal{T}$

Example

Λ: finite dimensional algebra over a field

$\mathcal{D} := \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda)$: the bounded derived category

$\Longrightarrow (\mathcal{D}_\Lambda^{\le 0}, \mathcal{D}_\Lambda^{\ge 0})$ is a bounded t-structure on $\mathcal{D}$.

$$\begin{aligned}\mathcal{D}_\Lambda^{\le 0} &:= \{X \in \mathcal{D} \mid H^{\forall i>0}(X) = 0\} \\ &= \{X \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(\Lambda, X[\forall i > 0]) = 0\}\end{aligned}$$

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Silting objects and t-structures

Λ: finite dimensional algebra over a field
$\mathcal{C} := \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$ and $\mathcal{D} := \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda)$

- Λ is a silting object of $\mathcal{C}$.
- $(\mathcal{D}^{\leq 0}_{\Lambda}, \mathcal{D}^{\geq 0}_{\Lambda})$ is a bounded t-structure on $\mathcal{D}$.

Theorem [Koenig-Yang (2014)]

- $^{\exists}$injection $\mathrm{silt}\mathcal{C} \to t\text{-}\mathrm{str}_{\mathrm{bd}}\mathcal{D} \quad (M \mapsto (\mathcal{D}^{\leq 0}_{M}, \mathcal{D}^{\geq 0}_{M}))$
- $\mathcal{D}^{\leq 0}_{M} \cap \mathcal{D}^{\geq 0}_{M} \simeq \mathsf{mod}\,\mathrm{End}_{\mathcal{D}}(M)$

$$\mathcal{D}^{\leq 0}_{M} := \{X \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(M, X[^{\forall}i > 0]) = 0\}$$
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ST-pair (S=silting object, T=t-structure)

$\mathcal{T}$: Hom-fintie Krull-Schmidt triangulated category
$\mathcal{C}, \mathcal{D}$: thick subcateogories of $\mathcal{T}$

Definition

$(\mathcal{C}, \mathcal{D})$: ST-pair inside $\mathcal{T}$ $:\Leftrightarrow$ $^{\exists}M$: silting object of $\mathcal{C}$ s.t.

(1) $(\mathcal{T}_M^{\leq 0}, \mathcal{T}_M^{\geq 0})$: t-structure on $\mathcal{T}$

(2) $\mathcal{T}_M^{\geq 0} \subseteq \mathcal{D}$

(3) $\mathcal{D} = \text{thick}\,\mathcal{T}_M^0$

The triple $(\mathcal{C}, \mathcal{D}, M)$ is called a ST-triple inside $\mathcal{T}$.

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Example of ST-pair

Example 1

Λ: finite dimensional algebra over a field k

$\Longrightarrow (\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda), \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda))$: ST-pair inside $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda)$

Example 2 [Amiot (2009), Kalck-Yang (2016)]

Γ: dg k-algebra satisfying

- $H^p(\Gamma) = 0$ $({}^\forall p > 0)$
- $H^0(\Gamma)$: finite dimensional
- $\mathsf{D}_{\mathrm{fd}}(\Gamma) \subseteq \mathsf{per}(\Gamma)$

$\Longrightarrow (\mathsf{per}(\Gamma), \mathsf{D}_{\mathrm{fd}}(\Gamma))$: ST-pair inside $\mathsf{per}(\Gamma)$

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Property of ST-pair

$(\mathcal{C},\mathcal{D})$: ST-pair inside $\mathcal{T}$ $:\Leftrightarrow$ ${}^{\exists}M$: silting object of $\mathcal{C}$ s.t.

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Properties

- (2) & (3) $\Rightarrow$ $(\mathcal{D}_M^{\leq 0}, \mathcal{D}_M^{\geq 0})$: bounded t-structure on $\mathcal{D}$
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- $\mathcal{D}_M^{\leq 0} \cap \mathcal{D}_M^{\geq 0} \simeq \text{mod}\,\text{End}_{\mathcal{T}}(M)$
- $M, N \in \mathcal{C}$: silting objects
  Then $(\mathcal{C},\mathcal{D},M)$: ST-triple $\Leftrightarrow$ $(\mathcal{C},\mathcal{D},N)$: ST-triple

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Main result 1

$(\mathcal{C}, \mathcal{D})$: ST-pair inside $\mathcal{T}$

Theorem

$^{\exists}$injection $\Psi : \mathrm{silt}\mathcal{C} \to t\text{-}\mathrm{str}_{\mathrm{bd}}\mathcal{D} \quad (M \mapsto (\mathcal{D}_M^{\leq 0}, \mathcal{D}_M^{\geq 0}))$

Question

When is Ψ a bijection?

Theorem [Keller-Vossieck (1988)]

Λ: path algebra of Dynkin type $\Longrightarrow$ Ψ: bijection

Λ: Kronecker algebra $\Longrightarrow$ Ψ: NOT bijective

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Silting-discrete triangulated category

$M \geq N :\Leftrightarrow \mathrm{Hom}_{\mathcal{T}}(M, N[{}^{\forall}i > 0]) = 0$

Proposition [Aihara-Iyama (2012)]

$(\mathrm{silt}\,\mathcal{T}, \geq)$ is a poset.

Definition [Aihara (2013)]

$\mathcal{T}$: silting-discrete $:\Leftrightarrow {}^{\forall}M \in \mathrm{silt}\,\mathcal{T}$ and ${}^{\forall}d \in \mathbb{Z}_{>0}$,

$d_M\text{-}\mathrm{silt}\,\mathcal{T} := \{N \in \mathrm{silt}\,\mathcal{T} \mid M \geq N \geq M[d-1]\}$: finite

$\mathcal{T}$: silting-discrete triangulated category

$\Rightarrow$ the poset $(\mathrm{silt}\,\mathcal{T}, \geq)$ has various good properties.

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$\mathcal{T}$: silting-discrete triangulated category
$\Rightarrow$ the poset $(\mathrm{silt}\,\mathcal{T}, \geq)$ has various good properties.

Main result 2

$(\mathcal{C}, \mathcal{D})$: ST-pair inside $\mathcal{T}$

Theorem

The following are equivalent:

(1) Ψ: bijection.

(2) $\mathcal{C}$: silting-discrete.

(3) The heart of any bounded t-structure on $\mathcal{D}$ has a projective generator.

$$\Psi : \mathrm{silt}\mathcal{C} \longrightarrow t\text{-}\mathrm{str}_{\mathrm{bd}}\mathcal{D} \quad (M \mapsto (\mathcal{D}_M^{\leq 0}, \mathcal{D}_M^{\geq 0}))$$

Application: Stability conditions

$(\mathcal{C}, \mathcal{D})$: ST-pair

Corollary [Qiu-Woolf (2014), BPP, PSZ, AMY]

$\mathcal{C}$: silting-discrete

$\Rightarrow$ the "stability manifold" $\mathrm{Stab}(\mathcal{D})$ is contractible.

BPP := Broomhead-Pauksztello-Ploog (2016)

PSZ := Pauksztello-Saorin-Zvonareva (2017)

Examples of silting-discrete algebras

Example

$\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$: silting-discrete if Λ is

- local algebras,
- representation-finite hereditary algebras,
- derived-discrete algebras,
- representation-finite symmetric algebras,
- generalized Brauer tree algebras,
- algebras of dihedral, semidihedral, quatenion type.

We want to construct various examples of silting-discrete triangulated categories.

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Give a criterion of silting-discrete triangulated categories by cluster theory.

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Calabi-Yau pair

$(\mathcal{C}, \mathcal{D})$: ST-pair (or $(\mathcal{C}, \mathcal{D}, M)$: ST-triple) inside $\mathcal{T}$

Definition [Iyama-Yang (2014)]

$(\mathcal{C}, \mathcal{D})$: d-Calabi-Yau (d-CY) pair $:\Leftrightarrow$

- $\mathcal{C} \supseteq \mathcal{D}$
- $^{\exists}$funct. isom.
  $\operatorname{Hom}_{\mathcal{T}}(X, Y) \simeq \mathbb{D} \operatorname{Hom}_{\mathcal{T}}(Y, X[d]) (X \in \mathcal{D}, Y \in \mathcal{C})$

Remark

$M, N \in \mathcal{C}$: silting objects
Then $(\mathcal{C}, \mathcal{D}, M)$: d-CY triple $\Leftrightarrow$ $(\mathcal{C}, \mathcal{D}, N)$: d-CY triple

Silting theory and Cluster theory

For simplicity, $d \geq 2$: integer.
$(\mathcal{C}, \mathcal{D})$: $(d+1)$-CY pair (or $(\mathcal{C}, \mathcal{D}, M)$: $(d+1)$-CY triple)

Theorem [Iyama-Yang (2014)]

(1) $\mathcal{U} := \mathcal{C}/\mathcal{D}$: d-Calabi-Yau triangulated category
(2) The canonical functor $\mathcal{C} \to \mathcal{U}$ induces an injection
$$\pi : d_M\text{-silt}\,\mathcal{C} \to d\text{-ctilt}\,\mathcal{U}.$$
(3) $d = 2 \Rightarrow$ the map π is bijective.

$U \in \mathcal{U}$: d-cluster-tilting (d-CT) object :⇔
add$U = \{X \in \mathcal{U} \mid \mathrm{Hom}_{\mathcal{U}}(U, X[i]) = 0 \ (1 \leq i \leq d-1)\}$

$d\text{-ctilt}\,\mathcal{U}$: the set of isoclasses of basic d-CT objects of $\mathcal{U}$

Cluster theory and silting theory

$(\mathcal{C}, \mathcal{D})$: $(d+1)$-CY pair

Theorem

$d\text{-ctilt}\,\mathcal{U}$: finite $\Longrightarrow$ $\mathcal{C}$: silting-discrete

Theorem

Assume $d = 2$.

The following are equivalent.

(1) $\mathcal{C}$: silting-discrete

(2) $2_M\text{-silt}\,\mathcal{C}$: finite set for all silting objects M

(2') $2_M\text{-silt}\,\mathcal{C}$: finite set for some silting object M

(3) $2\text{-ctilt}\,\mathcal{U}$: finite set

Cluster theory and silting theory

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Application: derived preprojective algebra

Q: finite (graded) quiver, $d \geq 1$
$\Gamma := \Gamma_{d+1}(Q)$: derived preprojective algebra
$H^0(\Gamma)$: finite dimensional

Lemma

$(\mathrm{per}(\Gamma), \mathrm{D}_{\mathrm{fd}}(\Gamma))$: $(d+1)$-CY pair

Theorem

The following are equivalent.

(1) $\mathrm{per}(\Gamma)$: silting-discrete

(2) Q: Dynkin

Application: complete Ginzburg dg algebra

(Q, W): quiver with potential
$\Gamma := \Gamma(Q, W)$: complete Ginzburg dg algebra
$H^0(\Gamma)$: finite dimensional

Lemma

$(\mathrm{per}(\Gamma), \mathrm{D}_{\mathrm{fd}}(\Gamma))$: 3-CY pair

Theorem

The following are equivalent.

(1) $\mathrm{per}(\Gamma)$: silting-discrete.
(2) Q is related to a Dynkin quiver by a finite sequence of quiver mutations.

Application: Stability conditions

$\Gamma \in \{\Gamma_{d+1}(Q), \Gamma(Q, W)\}$
$\mathrm{per}\,\Gamma$: silting-discrete

Corollary

The "stability manifold" $\mathrm{Stab}(\mathrm{D}_{\mathrm{fd}}(\Gamma))$ is contractible.

This result was a conjecture given by Yu Qiu (2011).