Lattices of torsion classes

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Joint with Osamu Iyama, Nathan Reading, Idun Reiten and Hugh Thomas.

A: f.d. algebra / field k.

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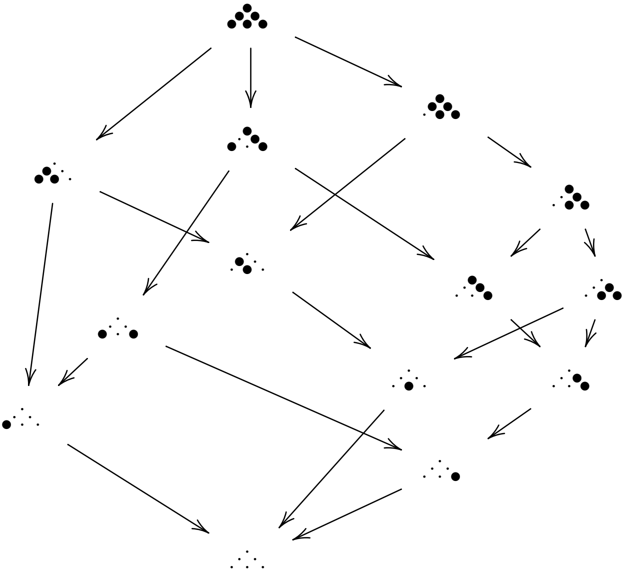

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From now on, $\# \operatorname{tors} A < \infty$.

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Theorem [Adachi-Iyama-Reiten]

1. Fac gives a bijection $\operatorname{s\tau-tilt} A \to \operatorname{tors} A$ (rem: $\# \operatorname{tors} A < \infty$).

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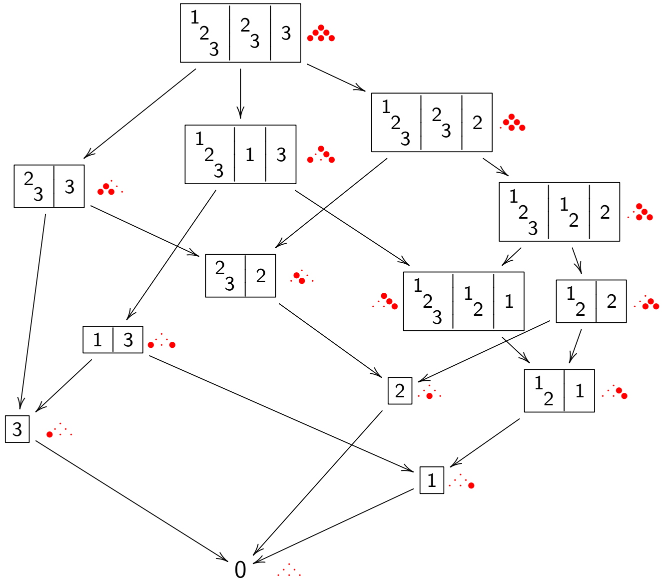

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2. $\exists T \to U$ in $\operatorname{Hasse}(\mathrm{s}\tau\text{-tilt}\, A) \Leftrightarrow T \cong T_0 \oplus X$, $U \cong T_0 \oplus X^*$, X and X^* indecomposable (or $X^* = 0$), $X \notin \operatorname{Fac} T_0$ and
$$\exists X \xrightarrow{u} U \to X^* \to 0,$$
u left min. $\operatorname{add}(T_0)$-approximation.

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Theorem [D-Iyama-Jasso, Asai, DIRRT]

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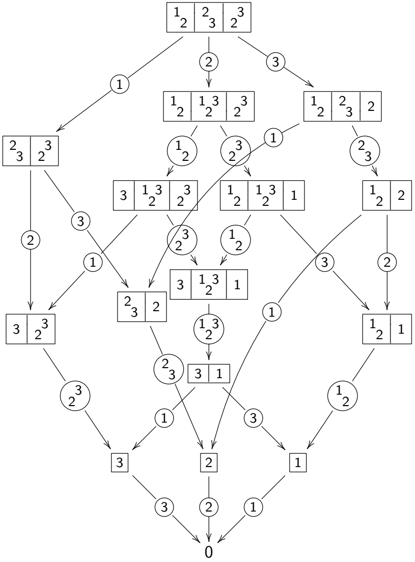
2. $q \mapsto S_q$ gives a bijection $\{q : \mathcal{T} \to \mathcal{U}, \mathcal{T}$ is $\vee$-irreducible$\} \to$ brick A.

$$\Lambda = \mathbb{C}\left(1 \xrightarrow{\alpha} 2 \underset{\beta^*}{\overset{\beta}{\rightleftarrows}} 3 \right) / (\alpha\beta, \beta\beta^*, \beta^*\beta)$$



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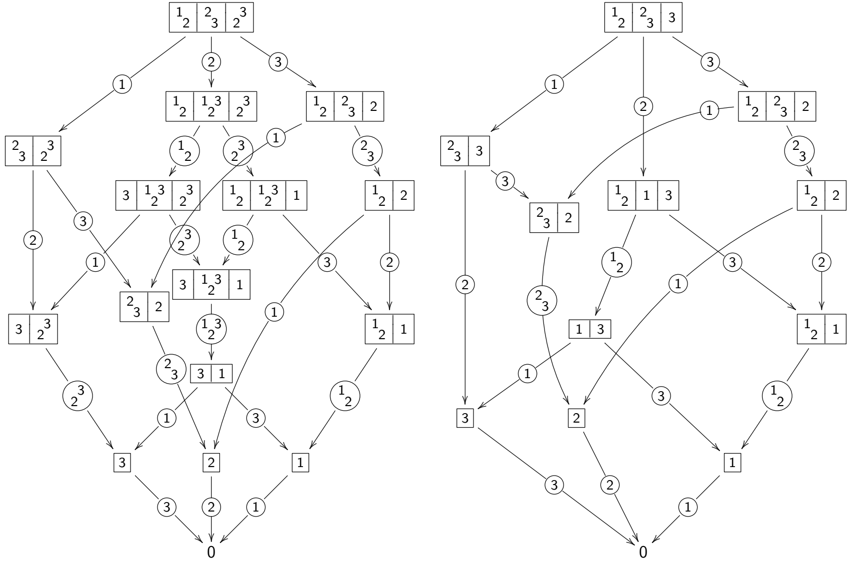

Forcing order and brick labelling

Forcing

$q, r \in \mathrm{Hasse}_1(\mathrm{tors}\, A)$
$q \rightsquigarrow r$ if $\forall \pi : \mathrm{tors}\, A \twoheadrightarrow L$ (lat. quot.), π contracts $q \Rightarrow \pi$ contracts r.

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Theorem [DIRRT]

$\rightsquigarrow$ on brick A is the transitive closure of

- $S \rightsquigarrow S'$ if $S' \in \mathrm{Filt}(S, S_2, \ldots, S_n) \setminus \mathrm{Filt}(S_2, \ldots, S_n)$
  for $\{S, S_2, S_3, \ldots, S_n\}$ semibrick.

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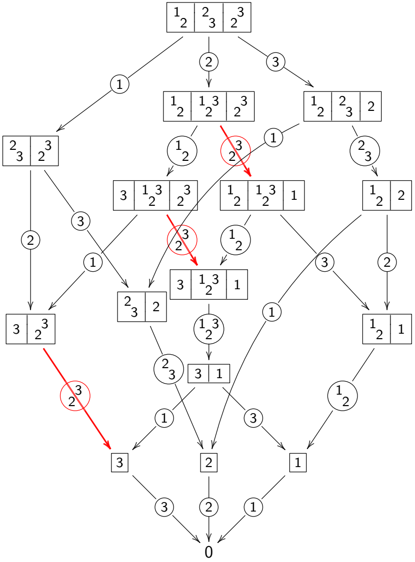

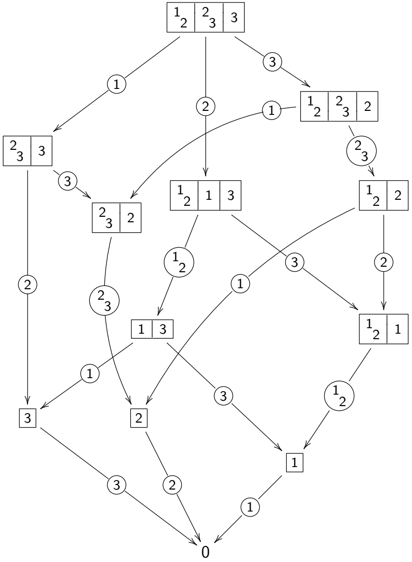

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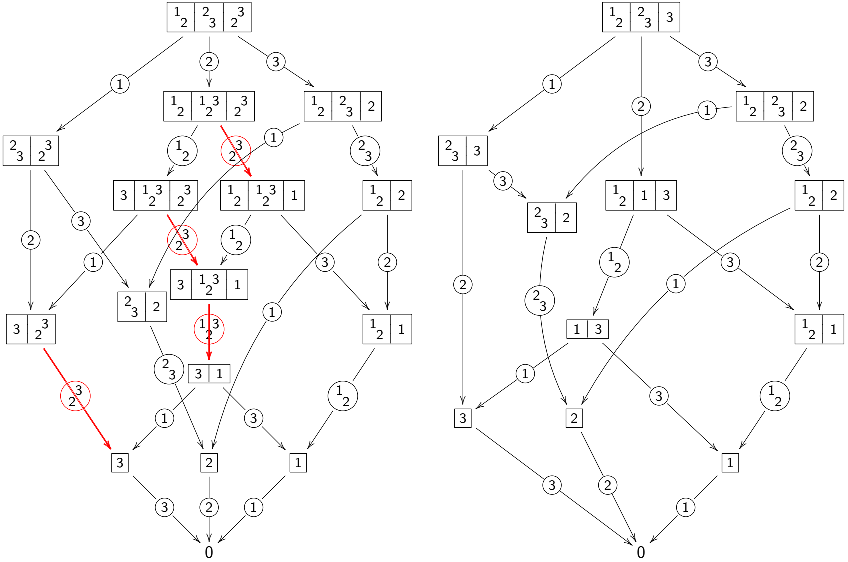

Algebraic quotients

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2. If q is not contracted by π then $S_q^{(A)} = S_q^{(B)}$.

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