On finitely graded Iwanaga-Gorenstein algebras and the stable categories of their (graded) Cohen-Macaulay modules

Hiroyuki Minamoto and Kota Yamaura

(Osaka Prefecture University,
University of Yamanashi)

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Setting of this talk

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- **a module := a right module**

Section 1. Self-injective dimension formula for trivial extension algebras

Remark 1.1

The contents of this section is taken from the paper "Homological dimension formulas for trivial extension algebras" arXiv 1710.01469

Section 1.1.
Trivial extension algebras

A trivial extension algebra $A = \Lambda \oplus C$

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$$(r, c)(s, d) := (rs, rd + cs)$$

for $r, s \in \Lambda$, $c, d \in C$.

A trivial extension algebra $A = \Lambda \oplus C$ has

Remark: a canonical grading of a trivial extension algebra

A trivial extension algebra $A = \Lambda \oplus C$ has the grading

$$\deg \Lambda = 0, \quad \deg C = 1.$$

Quasi-Veronese algebra (1/4)

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in the following way:

Quasi-Veronese algebra (3/4)

$$\nabla A := \begin{pmatrix} A_0 & A_1 & \cdots & A_{\ell-1} \\ 0 & A_0 & \cdots & A_{\ell-2} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_0 \end{pmatrix},$$

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In particular,
A : Iwanaga-Gorenstein $\Leftrightarrow$ $A^{[\ell]}$: Iwanaga-Gorenstein.

Section 1.2.
Self-injective dimension formula

Injective dimension of object of $\mathbf{D}(\mathbf{Mod}\,\Lambda)$ (1/2)

Definition 1 (Avramov-Foxby)

An object M of $\mathbf{D}(\mathbf{Mod}\,\Lambda)$ is said to have *injective dimension* at most n and is denoted as

$$\operatorname{id}_{\Lambda} M \leq n.$$

if it has an injective resolution I such that $I^m = 0$ for $m > n$.

$$\operatorname{id}_{\Lambda} M = n \Leftrightarrow \operatorname{id}_{\Lambda} M \leq n \text{ holds but } \operatorname{id}_{\Lambda} M \leq n-1 \text{ does not.}$$

Injective dimension of object of $\mathbf{D}(\mathbf{Mod}\,\Lambda)$ (2/2)

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(3) Θ_a *is an isomorphism for* $a \gg 0$.

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$$\alpha_r := \min\{a \geq 0 \mid \Theta_a \text{ is an isomorphism}\}.$$

The asid conditions and the asid number **(2/4)**

Let C be right asid and set $d := \mathrm{id}\, A$.

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Lemma 5

$\alpha_r = \max\{a \mid \exists n,\ \mathrm{soc}(\Omega^{-n}A)_{-a} \neq 0\} + 1$ *where* $a \geq -1$.

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A bimodule C is called asid if it is both left and right asid.

The asid conditions and the asid number **(4/4)**

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This class of IG-algebra $A = \mathbf{\Lambda} \oplus C$
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Section 1.3
The kernel of the canonical functor ϖ

To prove the self-injective dimension formula,
we make use of the grading of $A = \Lambda \oplus C$.
By the same method,
we obtain a description of
the kernel $\operatorname{Ker} \varpi$
of the canonical functor ϖ.

Let ϖ denotes the canonical functor

$$\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda) \hookrightarrow \mathsf{D}^{\mathrm{b}}(\mathsf{mod}^{\mathbb{Z}} A) \xrightarrow{\text{quotient}} \mathsf{Sing}^{\mathbb{Z}} A.$$

where

$$\mathsf{Sing}^{\mathbb{Z}} A := \mathsf{D}^{\mathrm{b}}(\mathsf{mod}^{\mathbb{Z}} A)/\mathsf{K}^{\mathrm{b}}(\mathsf{proj}^{\mathbb{Z}} A).$$

$$\mathrm{Ker}\,\varpi = \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda) \cap \mathsf{K}^{\mathrm{b}}(\mathsf{proj}^{\mathbb{Z}} A)$$

Assume that $\operatorname{pd} C_\Lambda < \infty$.
Then $- \otimes^{\mathbb{L}}_\Lambda C$ acts on $\mathsf{K}^{\mathrm{b}}(\operatorname{proj}\Lambda)$.

Proposition 1.6

$$\operatorname{Ker}\varpi = \bigcup_{a\geq 0} \operatorname{Ker}(- \otimes^{\mathbb{L}}_\Lambda C^a) \subset \mathsf{K}^{\mathrm{b}}(\operatorname{proj}\Lambda)$$

where we regard $- \otimes^{\mathbb{L}}_\Lambda C^a$ as an endofunctor of $\mathsf{K}^{\mathrm{b}}(\operatorname{proj}\Lambda)$.

Section 2.
On finitely graded IG-algebras and the stable categories of their (graded) CM-modules

Section 2.1.
(Graded) Iwanaga-Gorenstein algebras and (graded) Cohen-Macaulay modules

An algebra A is called

Iwanaga-Gorenstein(IG)

if it is Noetherian (on both sides) and

$$\operatorname*{id}_{A} A < \infty, \operatorname*{id}_{A^{\mathrm{op}}} A < \infty.$$

By Zaks' Theorem,

under Noetherian hypothesis,

the second condition is equivalent to

$$\operatorname*{id}_{A} A = \operatorname*{id}_{A^{\mathrm{op}}} A < \infty.$$

A graded algebra $A = \bigoplus_{i=0} A_i$ is called graded Iwanaga-Gorenstein(IG) if it is graded Noetherian (on both sides) and

$$\operatorname{gr.id}_{A} A < \infty, \operatorname{gr.id}_{A^{\mathrm{op}}} A < \infty.$$

By Zaks' Theorem, under graded Noetherian hypothesis, the second condition is equivalent to

$$\operatorname{gr.id}_{A} A = \operatorname{gr.id}_{A^{\mathrm{op}}} A < \infty.$$

A graded algebra $A = \bigoplus_{i \geq 0} A_i$ is graded IG if and only if it is IG as an ungraded algebra. Moreover,

$$\operatorname{gr.id}_A A \leq \operatorname{id}_A A \leq \operatorname{gr.id}_A A + 1.$$

The second inequality is due to M. Van den Bergh.

When $A = \bigoplus_{i=0}^{\ell} A_i$ is finitely graded, we have

$$\operatorname{gr.id}_A A = \operatorname{id}_A A.$$

Definition 6

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Graded CM-modules

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  (a triangulated category)

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$$\begin{array}{ccc}
\mathsf{K}^{\mathrm{ac}}(\mathsf{proj}^{\mathbb{Z}}\, A) & \xrightarrow[\sim]{\underline{\mathsf{Z}}^{\mathbf{0}}} & \underline{\mathsf{CM}}^{\mathbb{Z}}\, A \\
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Section 2.2.

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When is $A = \Lambda \oplus C$ IG?

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A categorical characterization (1/2)

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When $A = \Lambda \oplus C$ is IG $(1/4)$

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Theorem 8

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When $A = \Lambda \oplus C$ is IG (1/4)

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where ϖ denotes the canonical functor

$$\varpi : \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda) \hookrightarrow \mathsf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}} A) \to \mathsf{Sing}^{\mathbb{Z}} A$$

Theorem 9

If moreover $\mathbf{gldim}\, \boldsymbol{\Lambda} < \infty$,

When $A = \Lambda \oplus C$ is IG (2/4)

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\mathsf{T} \xrightarrow[\mathrm{in}|_{\mathsf{T}}]{\sim} & \mathfrak{D} & \xrightarrow[\pi|_{\mathfrak{D}}]{\sim} \mathsf{Sing}^{\mathbb{Z}} A
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$\varpi|_{\mathsf{T}} = \pi|_{\mathfrak{D}} \circ \mathbf{in}|_{\mathsf{T}}$.

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In particular, $\underline{\mathsf{CM}}^{\mathbb{Z}} A$ is realized as an admissible subcategory of $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} \Lambda)$.

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Corollary 10

$A = \bigoplus_{i=0}^{\ell} A_i$: a fin. dim. graded IG-algebra.

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$$\operatorname{rank} K_0(\underline{\mathsf{CM}}^{\mathbb{Z}} A) \leq \ell \#\{\text{simple } A\text{-modules}\}$$

When $A = \Lambda \oplus C$ is IG (4/4)

Remark 2.1

When $A = \Lambda \oplus C$ is IG (4/4)

Remark 2.1

In the case where Λ is IG,
we can obtain a similar commutative diagram
by introducing the notion of
locally perfect complexes .

Section 3. Applications

Section 3.1.
Two classes of IG algebras of finite CM-type

CM-version of Gabreial's Theorem in covering theory

Theorem 11 (MY-Yoshiwaki)

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$$\mathrm{ind}\,\mathbf{CM}^{\mathbb{Z}} A/(1) = \mathrm{ind}\,\mathbf{CM}\, A.$$

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The case $C = N \otimes_{\mathbf{k}} M$ (1/2)

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$\mathbb{R}\mathbf{Hom}(M, \mathbf{\Lambda}) = N[-p]$ *for some* $p \in \mathbb{N}$.

The case $C = N \otimes_{\mathsf{k}} M$ (2/2)

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Section 3.2. Classification of asid bimodule

Using the categorical characterization

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Using the categorical characterization obtained in Theorem 7, we obtain the complete list of asid modules C when Λ is the path algebra of A_2-quiver or A_3-quiver in the following strategy.

Step 1.

<u>Step 1.</u> Classify admissible subcategories T of $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda)$.

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Step 2.

Step 1. Classify admissible subcategories T of $\mathsf{K}^{\mathrm{b}}(\operatorname{proj}\Lambda)$.
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Step 2. For an admissible subcategory T,

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Step 1. **Classify admissible subcategories T of $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda)$.**
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Step 2. **For an admissible subcategory T, classify bimodules C such that the functor $- \otimes^{\mathbb{L}}_{\Lambda} C$ acts T as an equivalence and nilpotently acts on $\mathsf{T}^{\perp}$.**

The case $\Lambda = \mathsf{k}[1 \xleftarrow{\alpha} 2]$.

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$$\begin{array}{ccc} e_1 C e_1 & \xleftarrow{\cdot\,\alpha} & e_1 C e_2 \\ {\scriptstyle \alpha\,\cdot}\downarrow & & \downarrow{\scriptstyle \alpha\,\cdot} \\ e_2 C e_1 & \xleftarrow[\cdot\,\alpha]{} & e_2 C e_2 \end{array}$$

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e_i **: the idempotent of Λ corresponding to the vertex** i

The list of asid bimodule over $\mathbf{1} \leftarrow \mathbf{2}$ (1/3)

(I) $\quad T = \mathbf{D}^{\mathrm{b}}(\mathbf{mod}\,\Lambda)$

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(precisely the case $\alpha = 0$**.)**

The list of asid bimodule over $1 \leftarrow 2$ (1/3)

(I) $T = \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda)$
(precisely the case $\alpha = 0$.)

$$\Lambda = \begin{array}{ccc} \mathbf{k} & \leftarrow & 0 \\ \downarrow & & \downarrow \\ \mathbf{k} & \longleftarrow & \mathbf{k} \end{array},$$

(I) $T = \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$
(precisely the case $\alpha = 0$.)

$$\Lambda = \begin{array}{ccc} \mathbf{k} & \leftarrow & 0 \\ \downarrow & & \downarrow \\ \mathbf{k} & \leftarrow & \mathbf{k} \end{array}, \qquad \mathrm{D}(\Lambda) = \begin{array}{ccc} \mathbf{k} & \leftarrow & \mathbf{k} \\ \downarrow & & \downarrow \\ 0 & \leftarrow & \mathbf{k} \end{array}$$

(I) $T = \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$
(precisely the case $\alpha = 0$.)

$$\Lambda = \begin{array}{ccc} \mathbf{k} & \leftarrow & \mathbf{0} \\ \downarrow & & \downarrow \\ \mathbf{k} & \leftarrow & \mathbf{k} \end{array}, \qquad D(\Lambda) = \begin{array}{ccc} \mathbf{k} & \leftarrow & \mathbf{k} \\ \downarrow & & \downarrow \\ \mathbf{0} & \leftarrow & \mathbf{k} \end{array}$$

(II) $\mathsf{T} = \text{thick}\, P_1$

(I) $T = \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$
(precisely the case $\alpha = 0$.)

$$\Lambda = \begin{array}{ccc} k & \dashleftarrow & 0 \\ \downarrow & & \vdots \\ k & \longleftarrow & k \end{array}, \qquad D(\Lambda) = \begin{array}{ccc} k & \longleftarrow & k \\ \vdots & & \downarrow \\ 0 & \dashleftarrow & k \end{array}$$

(II) $\mathrm{T} = \mathrm{thick}\, P_1$

$$\Lambda e_1 \otimes_k e_1 \Lambda = \begin{array}{ccc} k & \dashleftarrow & 0 \\ \downarrow & & \vdots \\ k & \dashleftarrow & 0 \end{array}$$

(III) T = thick P_2

The list of asid bimodule over $\mathbf{1} \leftarrow \mathbf{2}$ (2/3)

(III) $\mathsf{T} = \mathsf{thick}\, P_2$

$$\Lambda e_2 \otimes_{\mathbf{k}} e_2\Lambda = \begin{array}{ccc} 0 & \leftarrow & 0 \\ \downarrow & & \downarrow \\ \mathbf{k} & \leftarrow & \mathbf{k} \end{array}$$

The list of asid bimodule over $\mathbf{1} \leftarrow \mathbf{2}$ (2/3)

(III) $\mathsf{T} = \text{thick}\, P_2$

$$\Lambda e_2 \otimes_{\mathbf{k}} e_2 \Lambda = \begin{array}{ccc} 0 & \leftarrow & 0 \\ \downarrow & & \downarrow \\ \mathbf{k} & \leftarrow & \mathbf{k} \end{array}$$

(IV) $\mathsf{T} = \text{thick}\, I_2$

The list of asid bimodule over $1 \leftarrow 2$ (2/3)

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$$\Lambda e_2 \otimes_{\mathsf{k}} e_2\Lambda = \begin{array}{ccc} 0 & \leftarrow & 0 \\ \downarrow & & \downarrow \\ \mathsf{k} & \leftarrow & \mathsf{k} \end{array}$$

(IV) $\mathsf{T} = \text{thick}\, I_2$

$$S_1^{\text{left}} \otimes_{\mathsf{k}} S_2^{\text{right}} = \begin{array}{ccc} 0 & \leftarrow & \mathsf{k} \\ \downarrow & & \downarrow \\ 0 & \leftarrow & 0 \end{array}$$

(V) $\mathsf{T} = \mathbf{0}$

(V) $\mathrm{T} = 0$
(precisely the case gldim $A < \infty$.)

(V) T = 0
(precisely the case gldim $A < \infty$.)

(V-1) $(\Lambda e_2 \otimes_{\mathbf{k}} e_1 \Lambda)^{\oplus n} = \begin{array}{ccc} \mathbf{0} & \leftarrow & \mathbf{0} \\ \downarrow & & \downarrow \\ \mathbf{k}^n & \leftarrow & \mathbf{0} \end{array}$

(V-2) $(S_1^{\text{left}} \otimes_{\mathbf{k}} e_2 \Lambda)^{\oplus n} = \begin{array}{ccc} \mathbf{k}^n & \leftarrow & \mathbf{k}^n \\ \downarrow & & \downarrow \\ \mathbf{0} & \leftarrow & \mathbf{0} \end{array}$

(V-3) $(\Lambda e_1 \otimes_{\mathbf{k}} S_2^{\text{right}})^{\oplus n} = \begin{array}{ccc} \mathbf{0} & \leftarrow & \mathbf{k}^n \\ \downarrow & & \downarrow \\ \mathbf{0} & \leftarrow & \mathbf{k}^n \end{array}$

The list of asid module C of $1 \leftarrow 2 \rightarrow 3$

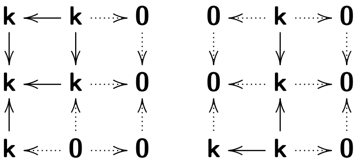
The list of asid module C of $1 \leftarrow 2 \rightarrow 3$ such that $\operatorname{gldim} A = \infty$.

(I) $\mathsf{T} = \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda)$
(precisely the case $\alpha = 0$.)

(I) $\mathsf{T} = \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda)$ (precisely the case $\alpha = 0$.)

$$
\begin{array}{ccccc}
k & \dashleftarrow & 0 & \dashrightarrow & 0 \\
\downarrow & & \dot\downarrow & & \dot\downarrow \\
k & \leftarrow & k & \rightarrow & k \\
\dot\uparrow & & \dot\uparrow & & \uparrow \\
0 & \dashleftarrow & 0 & \dashrightarrow & k
\end{array}
\qquad
\begin{array}{ccccc}
0 & \dashleftarrow & 0 & \dashrightarrow & k \\
\dot\downarrow & & \dot\downarrow & & \downarrow \\
k & \leftarrow & k & \rightarrow & k \\
\uparrow & & \dot\uparrow & & \dot\uparrow \\
k & \dashleftarrow & 0 & \dashrightarrow & 0
\end{array}
\qquad
\begin{array}{ccccc}
0 & \dashleftarrow & k & \rightarrow & k \\
\dot\downarrow & & \downarrow & & \dot\downarrow \\
0 & \dashleftarrow & k & \dashrightarrow & 0 \\
\dot\uparrow & & \uparrow & & \dot\uparrow \\
k & \leftarrow & k & \dashrightarrow & 0
\end{array}
\qquad
\begin{array}{ccccc}
k & \leftarrow & k & \dashrightarrow & 0 \\
\dot\downarrow & & \downarrow & & \dot\downarrow \\
0 & \dashleftarrow & k & \dashrightarrow & 0 \\
\dot\uparrow & & \uparrow & & \dot\uparrow \\
0 & \dashleftarrow & k & \rightarrow & k
\end{array}
$$

(II) $\mathsf{T} = \mathsf{thick}(P_1, I_1, I_2)$



The list for $1 \leftarrow 2 \rightarrow 3$, gldim $A = \infty$ (2/9)

(II) T = thick(P_1, I_1, I_2)

$$
\begin{array}{ccccc}
\mathbf{k} & \longleftarrow & \mathbf{k} & \dashrightarrow & 0 \\
\downarrow & & \downarrow & & \vdots \\
\mathbf{k} & \longleftarrow & \mathbf{k} & \dashrightarrow & 0 \\
\uparrow & & \vdots & & \vdots \\
\mathbf{k} & \dashleftarrow & 0 & \dashrightarrow & 0
\end{array}
\qquad
\begin{array}{ccccc}
0 & \dashleftarrow & \mathbf{k} & \dashrightarrow & 0 \\
\vdots & & \downarrow & & \vdots \\
0 & \dashleftarrow & \mathbf{k} & \dashrightarrow & 0 \\
\vdots & & \uparrow & & \vdots \\
\mathbf{k} & \longleftarrow & \mathbf{k} & \dashrightarrow & 0
\end{array}
$$

(III) T = thick(P_3, I_3, I_2)

$$
\begin{array}{ccccc}
0 & \dashleftarrow & 0 & \dashrightarrow & \mathbf{k} \\
\vdots & & \vdots & & \downarrow \\
0 & \dashleftarrow & \mathbf{k} & \longrightarrow & \mathbf{k} \\
\vdots & & \uparrow & & \uparrow \\
0 & \dashleftarrow & \mathbf{k} & \longrightarrow & \mathbf{k}
\end{array}
\qquad
\begin{array}{ccccc}
0 & \dashleftarrow & \mathbf{k} & \longrightarrow & \mathbf{k} \\
\vdots & & \downarrow & & \vdots \\
0 & \dashleftarrow & \mathbf{k} & \dashrightarrow & 0 \\
\vdots & & \uparrow & & \vdots \\
0 & \dashleftarrow & \mathbf{k} & \dashrightarrow & 0
\end{array}
$$

The list for $1 \leftarrow 2 \rightarrow 3$, gldim $A = \infty$ (3/9)

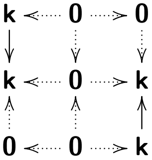
(IV) $\mathbf{T} = \mathbf{thick}(P_1, P_2, I_3)$

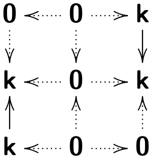
$$
\begin{array}{ccccc}
\mathbf{k} & \dashleftarrow & \mathbf{0} & \dashrightarrow & \mathbf{0} \\
\downarrow & & \vdots & & \vdots \\
\mathbf{k} & \leftarrow & \mathbf{k} & \rightarrow & \mathbf{k} \\
\vdots & & \vdots & & \vdots \\
\mathbf{0} & \dashleftarrow & \mathbf{0} & \dashrightarrow & \mathbf{0}
\end{array}
\qquad
\begin{array}{ccccc}
\mathbf{k} & \leftarrow & \mathbf{k} & \rightarrow & \mathbf{k} \\
\vdots & & \downarrow & & \downarrow \\
\mathbf{0} & \dashleftarrow & \mathbf{k} & \rightarrow & \mathbf{k} \\
\vdots & & \vdots & & \vdots \\
\mathbf{0} & \dashleftarrow & \mathbf{0} & \dashrightarrow & \mathbf{0}
\end{array}
$$

(V) $\mathbf{T} = \mathbf{thick}(P_3, P_2, I_1)$

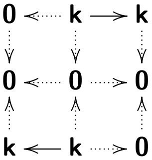
$$
\begin{array}{ccccc}
\mathbf{0} & \dashleftarrow & \mathbf{0} & \dashrightarrow & \mathbf{0} \\
\vdots & & \vdots & & \vdots \\
\mathbf{k} & \leftarrow & \mathbf{k} & \rightarrow & \mathbf{k} \\
\vdots & & \vdots & & \uparrow \\
\mathbf{0} & \dashleftarrow & \mathbf{0} & \dashrightarrow & \mathbf{k}
\end{array}
\qquad
\begin{array}{ccccc}
\mathbf{0} & \dashleftarrow & \mathbf{0} & \dashrightarrow & \mathbf{0} \\
\vdots & & \vdots & & \vdots \\
\mathbf{k} & \leftarrow & \mathbf{k} & \dashrightarrow & \mathbf{0} \\
\uparrow & & \uparrow & & \vdots \\
\mathbf{k} & \leftarrow & \mathbf{k} & \rightarrow & \mathbf{k}
\end{array}
$$

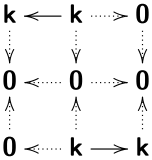
(VI) $\mathsf{T} = \mathsf{thick}(P_1, P_3)$





(VII) $\mathsf{T} = \mathsf{thick}(I_1, I_2)$





(VIII) T = thick P_3

$$
\begin{array}{ccccc}
0 & \dashleftarrow & 0 & \dashrightarrow & 0 \\
\vdots & & \vdots & & \vdots \\
\mathbf{k}^n & \dashleftarrow & 0 & \dashrightarrow & \mathbf{k} \\
\vdots & & \vdots & & \uparrow \\
0 & \dashleftarrow & 0 & \dashrightarrow & \mathbf{k}
\end{array}
\qquad
\begin{array}{ccccc}
\mathbf{k}^n & \leftarrow & \mathbf{k}^n & \dashrightarrow & 0 \\
\vdots & & \vdots & & \vdots \\
0 & \dashleftarrow & 0 & \dashrightarrow & \mathbf{k} \\
\vdots & & \vdots & & \uparrow \\
0 & \dashleftarrow & 0 & \dashrightarrow & \mathbf{k}
\end{array}
\qquad
\begin{array}{ccccc}
0 & \dashleftarrow & \mathbf{k}^n & \dashrightarrow & 0 \\
\vdots & & \downarrow & & \vdots \\
0 & \dashleftarrow & \mathbf{k}^n & \dashrightarrow & \mathbf{k} \\
\vdots & & \vdots & & \uparrow \\
0 & \dashleftarrow & 0 & \dashrightarrow & \mathbf{k}
\end{array}
$$

(IX) T = thick P_1

$$
\begin{array}{ccccc}
\mathbf{k} & \dashleftarrow & 0 & \dashrightarrow & 0 \\
\downarrow & & \vdots & & \vdots \\
\mathbf{k} & \dashleftarrow & 0 & \dashrightarrow & \mathbf{k}^n \\
\vdots & & \vdots & & \vdots \\
0 & \dashleftarrow & 0 & \dashrightarrow & 0
\end{array}
\qquad
\begin{array}{ccccc}
\mathbf{k} & \dashleftarrow & 0 & \dashrightarrow & 0 \\
\downarrow & & \vdots & & \vdots \\
\mathbf{k} & \dashleftarrow & 0 & \dashrightarrow & 0 \\
\vdots & & \vdots & & \vdots \\
0 & \dashleftarrow & \mathbf{k}^n & \rightarrow & \mathbf{k}^n
\end{array}
\qquad
\begin{array}{ccccc}
\mathbf{k} & \dashleftarrow & 0 & \dashrightarrow & 0 \\
\downarrow & & \vdots & & \vdots \\
\mathbf{k} & \dashleftarrow & \mathbf{k}^n & \dashrightarrow & 0 \\
\vdots & & \uparrow & & \vdots \\
0 & \dashleftarrow & \mathbf{k}^n & \dashrightarrow & 0
\end{array}
$$

(X) $\mathsf{T} = \text{thick } I_1$

$$\begin{array}{ccccc} 0 & \leftarrow & 0 & \rightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ k^n & \leftarrow & 0 & \rightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow \\ k^n & \leftarrow & 0 & \rightarrow & 0 \end{array} \oplus \begin{array}{ccccc} 0 & \leftarrow & 0 & \rightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \leftarrow & 0 & \rightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow \\ k & \leftarrow & k & \rightarrow & 0 \end{array}$$

$$\begin{array}{ccccc} 0 & \leftarrow & k^n & \rightarrow & k^n \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \leftarrow & k^n & \rightarrow & k^n \\ \uparrow & & \uparrow & & \uparrow \\ 0 & \leftarrow & k^n & \rightarrow & k^n \end{array} \oplus \begin{array}{ccccc} 0 & \leftarrow & 0 & \rightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \leftarrow & 0 & \rightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow \\ k & \leftarrow & k & \rightarrow & 0 \end{array}$$

$$\begin{array}{ccccc} k^n & \leftarrow & k^n & \rightarrow & k^n \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \leftarrow & 0 & \rightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow \\ k & \leftarrow & k & \rightarrow & 0 \end{array}$$

The list for $1 \leftarrow 2 \rightarrow 3$, gldim $A = \infty$ (7/9)

(XI) $\mathbf{T} = \text{thick } I_3$

$$
\begin{array}{ccccc}
0 & \dashleftarrow & 0 & \dashrightarrow & k^n \\
\vdots & & \vdots & & \downarrow \\
0 & \dashleftarrow & 0 & \dashrightarrow & k^n \\
\vdots & & \vdots & & \vdots \\
0 & \dashleftarrow & 0 & \dashrightarrow & 0
\end{array}
\oplus
\begin{array}{ccccc}
0 & \dashleftarrow & k & \longrightarrow & k \\
\vdots & & \vdots & & \vdots \\
0 & \dashleftarrow & 0 & \dashrightarrow & 0 \\
\vdots & & \vdots & & \vdots \\
0 & \dashleftarrow & 0 & \dashrightarrow & 0
\end{array}
$$

$$
\begin{array}{ccccc}
k^n & \leftarrow & k^n & \dashrightarrow & 0 \\
\downarrow & & \downarrow & & \vdots \\
k^n & \leftarrow & k^n & \dashrightarrow & 0 \\
\uparrow & & \uparrow & & \vdots \\
k^n & \leftarrow & k^n & \dashrightarrow & 0
\end{array}
\oplus
\begin{array}{ccccc}
0 & \dashleftarrow & k & \longrightarrow & k \\
\vdots & & \vdots & & \vdots \\
0 & \dashleftarrow & 0 & \dashrightarrow & 0 \\
\vdots & & \vdots & & \vdots \\
0 & \dashleftarrow & 0 & \dashrightarrow & 0
\end{array}
$$

$$
\begin{array}{ccccc}
0 & \dashleftarrow & k & \longrightarrow & k \\
\vdots & & \vdots & & \vdots \\
0 & \dashleftarrow & 0 & \dashrightarrow & 0 \\
\vdots & & \vdots & & \vdots \\
k^n & \leftarrow & k^n & \rightarrow & k^n
\end{array}
$$

(XII) T = thick P_2

$$\begin{array}{ccccc} 0 & \leftarrow & 0 & \rightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{k} & \leftarrow & \mathbf{k} & \rightarrow & \mathbf{k} \\ \uparrow & & \uparrow & & \uparrow \\ \mathbf{k}^n & \leftarrow & 0 & \rightarrow & 0 \end{array} \qquad \begin{array}{ccccc} 0 & \leftarrow & 0 & \rightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \leftarrow & 0 & \rightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow \\ \mathbf{k}^n & \leftarrow & 0 & \rightarrow & 0 \end{array} \oplus \begin{array}{ccccc} 0 & \leftarrow & 0 & \rightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{k} & \leftarrow & \mathbf{k} & \rightarrow & \mathbf{k} \\ \uparrow & & \uparrow & & \uparrow \\ \mathbf{k} & \leftarrow & 0 & \rightarrow & 0 \end{array}$$

$$\begin{array}{ccccc} 0 & \leftarrow & 0 & \rightarrow & \mathbf{k}^n \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{k} & \leftarrow & \mathbf{k} & \rightarrow & \mathbf{k} \\ \uparrow & & \uparrow & & \uparrow \\ 0 & \leftarrow & 0 & \rightarrow & 0 \end{array} \qquad \begin{array}{ccccc} 0 & \leftarrow & 0 & \rightarrow & \mathbf{k}^n \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \leftarrow & 0 & \rightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow \\ 0 & \leftarrow & 0 & \rightarrow & 0 \end{array} \oplus \begin{array}{ccccc} 0 & \leftarrow & 0 & \rightarrow & \mathbf{k} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{k} & \leftarrow & \mathbf{k} & \rightarrow & \mathbf{k} \\ \uparrow & & \uparrow & & \uparrow \\ 0 & \leftarrow & 0 & \rightarrow & 0 \end{array}$$

(XIII) $\mathsf{T} = \mathrm{thick}\, I_2$

$$
\begin{array}{ccccc}
\mathbf{0} & \dashleftarrow & \mathbf{k}^n & \rightarrow & \mathbf{k}^n \\
\vdots & & \downarrow & & \downarrow \\
\mathbf{0} & \dashleftarrow & \mathbf{k}^n & \rightarrow & \mathbf{k}^n \\
\vdots & & \vdots & & \vdots \\
\mathbf{0} & \dashleftarrow & \mathbf{0} & \dashrightarrow & \mathbf{0}
\end{array}
\oplus
\begin{array}{ccccc}
\mathbf{0} & \dashleftarrow & \mathbf{k} & \dashrightarrow & \mathbf{0} \\
\vdots & & \downarrow & & \vdots \\
\mathbf{0} & \dashleftarrow & \mathbf{k} & \dashrightarrow & \mathbf{0} \\
\vdots & & \uparrow & & \vdots \\
\mathbf{0} & \dashleftarrow & \mathbf{k} & \dashrightarrow & \mathbf{0}
\end{array}
\qquad
\begin{array}{ccccc}
\mathbf{0} & \dashleftarrow & \mathbf{0} & \dashrightarrow & \mathbf{0} \\
\vdots & & \vdots & & \vdots \\
\mathbf{k}^n & \leftarrow & \mathbf{k}^n & \dashrightarrow & \mathbf{0} \\
\uparrow & & \uparrow & & \vdots \\
\mathbf{k}^n & \leftarrow & \mathbf{k}^n & \dashrightarrow & \mathbf{0}
\end{array}
\oplus
\begin{array}{ccccc}
\mathbf{0} & \dashleftarrow & \mathbf{k} & \dashrightarrow & \mathbf{0} \\
\vdots & & \downarrow & & \vdots \\
\mathbf{0} & \dashleftarrow & \mathbf{k} & \dashrightarrow & \mathbf{0} \\
\vdots & & \uparrow & & \vdots \\
\mathbf{0} & \dashleftarrow & \mathbf{k} & \dashrightarrow & \mathbf{0}
\end{array}
$$

The last quivers

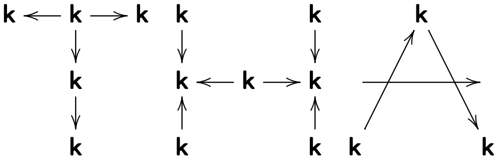
The last quivers

$$
\begin{array}{ccccc}
\mathbf{k} & \longleftarrow & \mathbf{k} & \longrightarrow & \mathbf{k} \\
 & & \downarrow & & \\
 & & \mathbf{k} & & \\
 & & \downarrow & & \\
 & & \mathbf{k} & &
\end{array}
$$

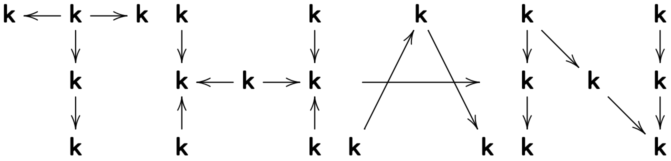
The last quivers

$$
\begin{array}{ccccc}
\mathbf{k} & \longleftarrow & \mathbf{k} & \longrightarrow & \mathbf{k} \\
 & & \downarrow & & \\
 & & \mathbf{k} & & \\
 & & \downarrow & & \\
 & & \mathbf{k} & &
\end{array}
\qquad
\begin{array}{ccccc}
\mathbf{k} & & & & \mathbf{k} \\
\downarrow & & & & \downarrow \\
\mathbf{k} & \longleftarrow & \mathbf{k} & \longrightarrow & \mathbf{k} \\
\uparrow & & & & \uparrow \\
\mathbf{k} & & & & \mathbf{k}
\end{array}
$$

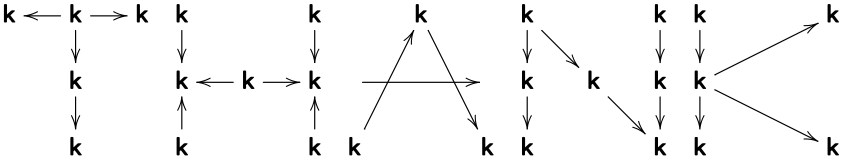
The last quivers



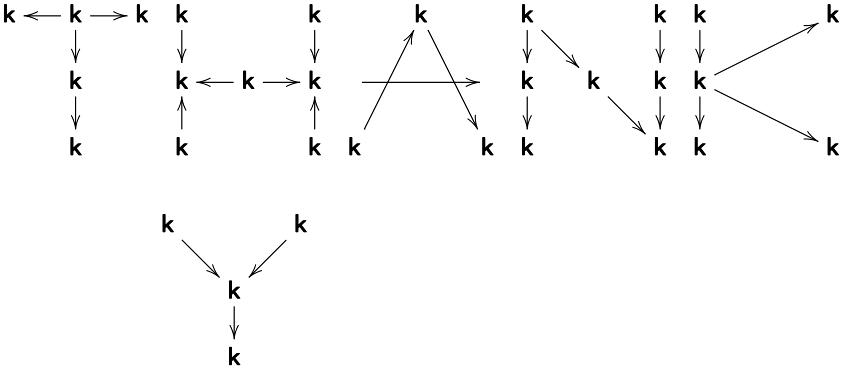
The last quivers



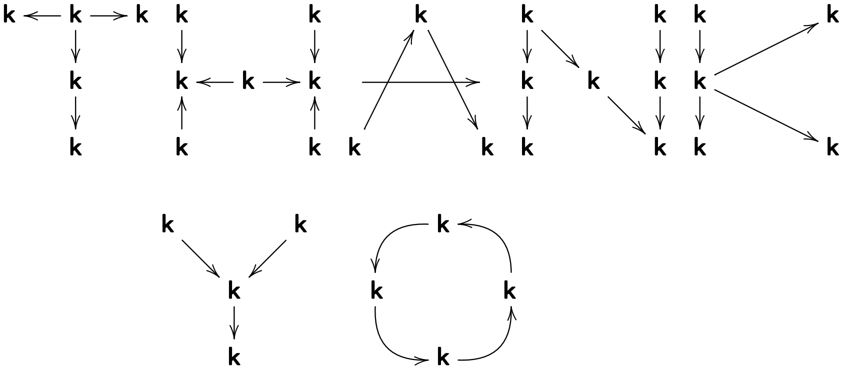
The last quivers



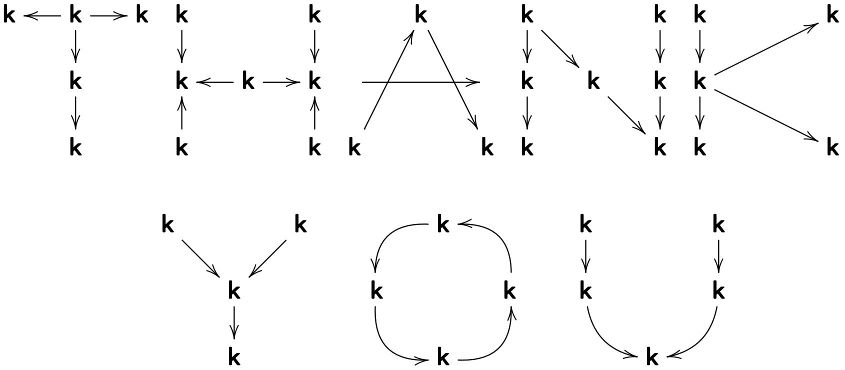
The last quivers



The last quivers



The last quivers



Thank you

Thank you
ありがとうございました

Thank you

ありがとうございました

Danke schön

Thank you
ありがとうございました
Danke schön
Merci beaucoup

Thank you

ありがとうございました

Danke schön

Merci beaucoup

Tack så mycket

Thank you

ありがとうございました

Danke schön

Merci beaucoup

Tack så mycket

謝謝

Thank you
ありがとうございました
Danke schön
Merci beaucoup
Tack så mycket
謝謝
Kamsahamnida

Thank you
ありがとうございました
Danke schön
Merci beaucoup
Tack så mycket
謝謝
Kamsahamnida
Cam on nhieu

Thank you

ありがとうございました

Danke schön

Merci beaucoup

Tack så mycket

謝謝

Kamsahamnida

Cam on nhieu

dhônyôbad

Thank you

ありがとうございました

Danke schön

Merci beaucoup

Tack så mycket

謝謝

Kamsahamnida

Cam on nhieu

dhônyôbad

thanks a lot!!

Thank you

ありがとうございました

Danke schön

Merci beaucoup

Tack så mycket

謝謝

Kamsahamnida

Cam on nhieu

dhônyôbad

thanks a lot!!(literally)

A question for the audience

Problem 14

A question for the audience

Problem 14

Naming.

A question for the audience

Problem 14

Naming. Is "asid" a good name?