Two-term silting complexes over radical square zero algebras

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Contents

- Λ : a finite dimensional algebra over a field $k = \overline{k}$.
- $\mathcal{T} := \mathsf{K}^{\mathrm{b}}(\operatorname{proj}\Lambda)$: the homotopy category of bounded complexes of $\operatorname{proj}\Lambda$.

Definition (Silting complex)

Let T be a complex in $\mathcal{T}$. Then T is said to be *silting* if

1. $\mathrm{Hom}_{\mathcal{T}}(T, T[i]) = 0$ for $i > 0$, and
2. $\mathcal{T} = \operatorname{thick} T$.

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2. $\mathcal{T} = \operatorname{thick} T$.

- $2\text{-silt}\,\Lambda$: the set of isomorphism classes of basic two-term silting complexes for Λ, where a complex T is *two-term* if $T \cong (T^{-1} \xrightarrow{d_T} T^0)$

1. Introduction
2. Main result
3. Symmetric radical cube zero algebras

Main result

From now on

- Suppose Λ is an algebra with radical square zero, i.e. ${J_\Lambda}^2 = 0$ where J_Λ is the Jacobson radical of Λ.
- $Q := (Q_0, Q_1)$: the (ordinary) quiver of Λ.
- $\epsilon : Q_0 \to \{+, -\}$: a map called *signature* on Q.

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Define a subquiver Q_ϵ of Q as

- $(Q_\epsilon)_0 := Q_0$,
- $(Q_\epsilon)_1 := \{i \to j \text{ in } Q \mid \epsilon(i) = + \text{ and } \epsilon(j) = -\}$.

$\rightsquigarrow$ Q_ϵ is a bipartite quiver, i.e. every vertex is either a source or a sink.

Main result

- $2\text{-silt}_\epsilon \Lambda$: a subset of $2\text{-silt}\,\Lambda$ consisting of all complexes $T = (T^{-1} \xrightarrow{d_T} T^0)$ such that

$$T^{-1} \in \mathrm{add}\Big(\bigoplus_{\epsilon(j)=-} P(j)\Big) \text{ and } T^0 \in \mathrm{add}\Big(\bigoplus_{\epsilon(i)=+} P(i)\Big).$$

Proposition

$$2\text{-silt}\,\Lambda = \coprod_{\epsilon:\,\mathrm{sgn\ on}\ Q} 2\text{-silt}_\epsilon \Lambda.$$

Main result

Theorem (A.)

For each signature ϵ on Q, there are bijections between:

(1) $2\text{-silt}_\epsilon \Lambda$,

(2) $2\text{-silt}_\epsilon kQ_\epsilon$,

(3) $\text{tilt}\, kQ_\epsilon^{\text{op}}$: *the set of isomorphism classes of basic tilting modules over kQ_ϵ^{op},*

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which induce isomorphisms of partially ordered sets. Therefore, we have a bijection

$$2\text{-silt}\,\Lambda \overset{1-1}{\longleftrightarrow} \coprod_{\epsilon:\,\text{sgn on } Q} \text{tilt}\, kQ_\epsilon^{\text{op}}.$$

Corollary (Adachi '16, A.)

The set $2\text{-silt}\,\Lambda$ is finite if and only if Q_ϵ is a disjoint union of Dynkin quivers for every signature ϵ on Q. In this situation, we have

$$\#\,2\text{-silt}\,\Lambda = \sum_{\epsilon:\,\text{sgn on } Q} \#\,\text{tilt}\,kQ_\epsilon^{\mathrm{op}}$$

Remark $\#\,\text{tilt}\,kQ_\epsilon^{\mathrm{op}}$ for Dynkin type is given by

- $\mathbb{A}_n$: $C_n = \frac{1}{n+1}\binom{2n}{n}$: the Catalan numbers,
- $\mathbb{D}_n$: $\frac{3n-4}{2n-2}\binom{2n-2}{n-2}$,
- $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$: $418, 2431, 17342$ respectively.

Example

$$Q: \begin{tikzcd} 1 \arrow[r] & 2 \arrow[r, shift left] & 3 \arrow[l, shift left] \arrow[ll, bend right] \end{tikzcd}, \ \Lambda = kQ/I : \text{radical square zero.}$$

$$\begin{array}{l} \epsilon(1) = \epsilon(3) = + \\ \epsilon(2) = - \end{array} \Longrightarrow \ Q_\epsilon: \ 1 \longrightarrow 2 \longleftarrow 3$$

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$$\begin{array}{l} \epsilon(1)=\epsilon(3)=+ \\ \epsilon(2)=- \end{array} \implies Q_\epsilon:\ \begin{tikzcd} 1 \arrow[r] & 2 & 3 \arrow[l] \end{tikzcd}$$

<u>2-silt$_\epsilon$ Λ</u>

$$\begin{tikzcd}[column sep=tiny]
& \boxed{\begin{array}{c} 0 \to P(1) \\ P(2) \to P(1)\oplus P(3) \\ 0 \to P(3) \end{array}} \arrow[dl] \arrow[dr] & \\
\boxed{\begin{array}{c} P(2) \to P(3) \\ P(2) \to P(1)\oplus P(3) \\ 0 \to P(3) \end{array}} \arrow[dr] & & \boxed{\begin{array}{c} 0 \to P(1) \\ P(2) \to P(1)\oplus P(3) \\ P(2) \to P(1) \end{array}} \arrow[dl] \\
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<u>tilt $kQ_\epsilon^{\mathrm{op}}$</u>

$$\begin{tikzcd}[column sep=tiny]
& \left(1,\ \begin{smallmatrix}2\\13\end{smallmatrix},\ 3\right) \arrow[dl] \arrow[dr] & \\
\left(\begin{smallmatrix}2\\3\end{smallmatrix},\ \begin{smallmatrix}2\\13\end{smallmatrix},\ 3\right) \arrow[dr] & & \left(1,\ \begin{smallmatrix}2\\13\end{smallmatrix},\ \begin{smallmatrix}2\\1\end{smallmatrix}\right) \arrow[dl] \\
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$$Q: \begin{tikzcd} 1 \arrow[r] & 2 \arrow[r, shift left] & 3 \arrow[l, shift left] \arrow[ll, bend right] \end{tikzcd}, \quad \Lambda = kQ/I : \text{radical square zero.}$$

$$\begin{array}{l} \epsilon(1) = \epsilon(3) = + \\ \epsilon(2) = - \end{array} \implies Q_\epsilon : \begin{tikzcd} 1 \arrow[r] & 2 & 3 \arrow[l] \end{tikzcd}$$

2-silt$_\epsilon\,\Lambda$

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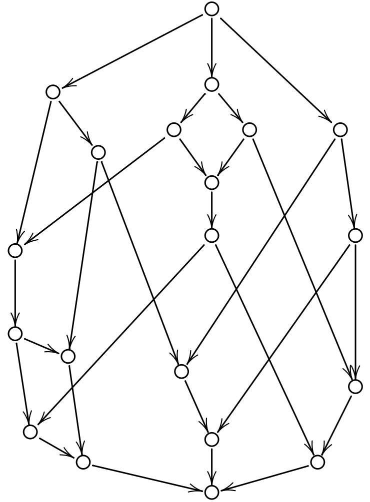
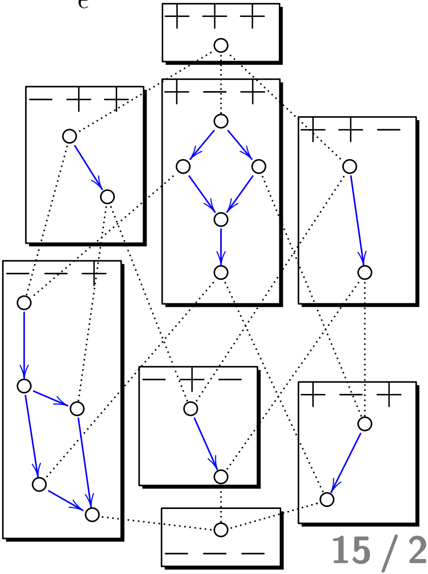
\end{tikzcd}$$

$$Q: \begin{tikzcd} 1 \arrow[r] & 2 \arrow[r, shift left] & 3 \arrow[l, shift left] \arrow[ll, bend right] \end{tikzcd}$$

$$\mathrm{Q}(2\text{-silt}\,\Lambda)$$



$$\coprod_{\epsilon} \mathrm{Q}(\mathrm{tilt}\; kQ_{\epsilon}^{\mathrm{op}})$$



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Symmetric radical cube zero algebras

Proposition (Adachi, Eisele-Janssens-Raedschelders)

Let Γ be a symmetric radical cube zero algebra. Then $\overline{\Gamma} := \Gamma/\mathrm{soc}\Gamma$ is radical square zero and we have an isomorphism of partially ordered set:

$$2\text{-tilt}\,\Gamma \cong 2\text{-silt}\,\overline{\Gamma}.$$

$\rightsquigarrow$ We can also apply our results for symmetric radical cube zero algebras!

Definition (Brauer line algebras)

A *multiplicity-free* *Brauer line algebra* $\Gamma_n := kQ_\Gamma / I_\Gamma$ *with* n *vertices* is defined by the following quiver and relations:

$$Q_\Gamma: \; 1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} 3 \underset{\beta_3}{\overset{\alpha_3}{\rightleftarrows}} \quad \cdots \quad \underset{\beta_{n-2}}{\overset{\alpha_{n-2}}{\rightleftarrows}} n-1 \underset{\beta_{n-1}}{\overset{\alpha_{n-1}}{\rightleftarrows}} n,$$

$$I_\Gamma = \langle \alpha_1\beta_1\alpha_1,\ \alpha_i\alpha_{i+1},\ \beta_{i+1}\beta_i,\ \beta_i\alpha_i - \alpha_{i+1}\beta_{i+1},\ \beta_{n-1}\alpha_{n-1}\beta_{n-1} \mid i = 1, \ldots, n-2 \rangle.$$

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Theorem (A)

Let Γ_n be a Brauer line algebra with n vertices, then we have

$$\#\,\text{2-tilt}\,\Gamma_n = \#\,\text{2-silt}\,\overline{\Gamma_n} = \binom{2n}{n}.$$

Summery

For radical square zero algebras:

- Two-term silting complexes correspond to tilting modules over certain path algebras.
- We have an isomorphism of posets for each component.
  - Can we recover the whole of $\mathrm{Q}(2\text{-silt}\,\Lambda)$?
- We can apply the results for symmetric radical cube zero algebras.
  - For a Brauer line algebra Γ_n, we have $\#\,2\text{-tilt}\,\Gamma_n = \binom{2n}{n}$.
  - Can we calculate $\mathrm{End}(T)$ by using this result?

References

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Thank you for your attention!