On the vanishing of self extensions over Cohen-Macaulay local rings

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$M \in \operatorname{mod} R$,
M is locally free on X^n $\iff$ $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \mathsf{X}^n$.

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Theorem 2 (Huneke-Leuschke (2004), A (2009))

Let R be a Gorenstein normal domain and $M \in \operatorname{mod} R$.
If $\operatorname{Ext}_R^{>0}(M, M \oplus R) = 0$, then M is free.

Theorem 3 (A)

Let R be a Gorenstein local ring of dimension $d > 1$ and $M \in \operatorname{mod} R$. Assume

(1) *M is maximal Cohen-Macaulay,*

(2) *M is locally free on $X^{d-1}(R)$,*

(3) $\operatorname{Ext}_R^{d-1}(M, M) = 0$.

Then, M is free.

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Theorem 4 (Ono-Yoshino (2017))

Let R be a Gorenstein local ring of dimension $d > 2$ and $M \in \operatorname{mod} R$. Assume

(1) M *is maximal Cohen-Macaulay,*

(2) M *is locally free on* $X^{d-2}(R)$*,*

(3) $\operatorname{Ext}_R^{d-1}(M, M) = \operatorname{Ext}_R^{d-2}(M, M) = 0$.

Then, M is free.

Question 5

Let R be a Gorenstein local ring of dimension d and $M \in \operatorname{mod} R$.
Let $0 < n < d$. Assume

(1) M is maximal Cohen-Macaulay,

(2) M is locally free on $X^{d-n}(R)$,

(3) $\operatorname{Ext}_R^i(M, M) = 0$ for all $d - n \leq i \leq d - 1$.

Then, is M free?

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Theorem 6 (A-Celikbas-Sadeghi-Takahashi(2017))

Let R be a Cohen-Macaulay local ring of dimension d with canonical module ω. Let $M \in \operatorname{mod} R$ and $0 < n < d$. Assume

(1) *M satisfies (S_2) and M^* is maximal Cohen-Macaulay,*

(2) *$\operatorname{pd}_{R_\mathfrak{p}} M_\mathfrak{p} < \infty$ for all $\mathfrak{p} \in X^{d-n}(R)$,*

(3) *$\operatorname{Ext}_R^i(M, (M^*)^\dagger) = 0$ for all $d - n \leq i \leq d - 1$.*

Then, M is free. Here, $(-)^* := \operatorname{Hom}_R(-, R)$, $(-)^\dagger := \operatorname{Hom}_R(-, \omega)$

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Claim. $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \operatorname{Spec} R$.

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Proof

Claim. $M_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module for all $\mathfrak{p} \in \operatorname{Spec} R$.
Induction on $\operatorname{ht} \mathfrak{p} =: t$. If $t \leq d - n$, then $M_\mathfrak{p}$ is free by (2).
Suppose $d - n < t (\leq d)$. Localize at $\mathfrak{p}$,
$R_\mathfrak{p}$ is Gorenstein local ring of dimension t.
$M_\mathfrak{p}$ is a MCM $R_\mathfrak{p}$-module which is locally free on $X^{t-1}(R_\mathfrak{p})$.
$\operatorname{Ext}^{t-1}_{R_\mathfrak{p}}(M_\mathfrak{p}, M_\mathfrak{p}) = \operatorname{Ext}^{t-1}_R(M, M)_\mathfrak{p} = 0$ $(d - n \leq t - 1 \leq d - 1)$.

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Then, $M_\mathfrak{p}$ is free by Theorem 3. □

Thank you!