One delta invariants of certain ideals

Toshinori Kobayashi

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The delta invariants of modules

- $(R, \mathfrak{m})$: commutative Cohen-Macaulay local ring having a canonical module and infinite residue field $R/\mathfrak{m}$.
- $\mathsf{CM}(R)$: the category of maximal Cohen-Macaulay R-modules.

Definition

Let M be a finitely generated R-module, $i \geq 0$ and $X_M \to M$ be a minimal right $\mathsf{CM}(R)$-cover of M. Then we define

- $\delta(M)$ to be the rank of free summand of X_M.
- $\delta^i(M) := \delta(\Omega^i M)$, where $\Omega^i M$ is the i-th syzygy module of M in the minimal free resolution.

Theorem (Auslander, Yoshino)

Let $d > 0$ be the Krull dimension of R. Consider the following conditions.

(a) R is a regular local ring.

(b) There exists $n \geq 0$ such that $\delta^n(R/\mathfrak{m}) > 0$.

(c) There exist $n > 0$ and $l > 0$ such that $\delta^n(R/\mathfrak{m}^l) > 0$.

Then, the implications $(\mathrm{a}) \Leftrightarrow (\mathrm{b}) \Rightarrow (\mathrm{c})$ hold. The implication $(\mathrm{c}) \Rightarrow (\mathrm{a})$ holds if depth $\mathrm{gr}_{\mathfrak{m}}(R) \geq d - 1$.

Here we denote by $\mathrm{gr}_I(R)$ the associated graded ring of R with respect to an ideal I of R.

Main Theorem

Theorem (K.)

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with a canonical module ω, having infinite residue field k and Krull dimension $d > 0$. Let I be an $\mathfrak{m}$-primary ideal of R such that I/I^2 is a free R/I-module. Consider the following conditions.

(a) $\delta(R/I) > 0$.

(b) I is a parameter ideal of R.

(c) There exists $n \geq 0$ such that $\delta^n(R/I) > 0$.

(d) There exist $n > 0$ and $l > 0$ such that $\delta^n(R/I^l) > 0$.

Then, the implications $(\mathrm{a}) \Rightarrow (\mathrm{b}) \Leftrightarrow (\mathrm{c}) \Rightarrow (\mathrm{d})$ hold. The implication $(\mathrm{d}) \Rightarrow (\mathrm{c})$ holds if $\operatorname{depth} \mathrm{gr}_I(R) \geq d-1$ and I^i/I^{i+1} is a free R/I-module for any $i > 0$. The implication $(\mathrm{b}) \Rightarrow (\mathrm{a})$ holds if $I \subset \mathrm{tr}(\omega)$.

Lemma (Yoshida)

Let M be a finitely generated R-module and $x \in \mathfrak{m}$ be a regular element on M and R. If $0 \to Y \to X \to M \to 0$ is a minimal Cohen-Macaulay approximation of M, then

$$0 \to Y/xY \to X/xX \to M/xM \to 0$$

is a minimal Cohen-Macaulay approximation of M/xM over $R/(x)$. In particular, it holds that $\delta_R(M) \leq \delta_{R/(x)}(M/xM)$.

Ulrich ideal

Definition

We say that an $\mathfrak{m}$-primary ideal I is an *Ulrich ideal* of R if it satisfies the follwing.

(1) $\mathrm{gr}_I(R)$ is a Cohen-Macaulay ring with $a(\mathrm{gr}_I(R)) \leq 1 - d$.

(2) I/I^2 is a free R/I-module.

Ulrich ideals satisfy the assumption of the main theorem.

Lemma

Let I be an Ulrich ideal of R. Then I^l/I^{l+1} is a free R/I-module for any $l \geq 1$.

Ulrich ideal

- index$(R) := \inf\{l \mid \delta(R/\mathfrak{m}^l) = 1\}$.
- index$(R) < \infty$ if and only if R is Gorenstein on the punctured spectrum.

Corollary

Assume that R is Gorenstein on the punctured spectrum.
I: Urlich ideal that is not a parameter ideal $\implies I \not\subset \mathfrak{m}^{\text{index}(R)}$.

References

[1] M. Auslander, S. Ding, Ø. Solberg, Liftings and weak liftings of modules, *J. Algebra* 156 (1993), 273–317.

[2] S. Goto, K.Ozeki, R. Takahashi, K. Watanabe, K. Yoshida, Ulrich ideals and modules, *Math. Proc. Cambridge Philos. Soc.* 156 (2014), no. 1, 137–166.

[3] T. Kobayashi, On delta invariants and indices of ideals, arXiv:1705.05042.

[4] K.Yoshida, A note on minimal Cohen-Macaulay approximations, Communications in Algebra 24(1) (1996), 235–246.

[5] Y. Yoshino, On the higher delta invariants of a Gorenstein local ring, *Proc. Amer. Math. Soc.* 124 (1996), 2641–2647.