The Krull dimension of composite power series rings over valuation rings

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Definition

Let $f : A \to B$ be a ring homomorphism and J an ideal of B. The subring $A \bowtie J$ of $A \times B$ is defined as follows:

$$A \bowtie^f J = \{(a, f(a) + j) \mid a \in A \text{ and } j \in J\}$$

We call the ring $A \bowtie^f J$ the amalgamation of A with B along J with respect to f.

$$\begin{array}{ccc} A \bowtie^f J & \xrightarrow{p_A} & A \\ {\scriptstyle p_B}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ B & \xrightarrow{\pi} & B/J \end{array} \qquad (1)$$

where $\pi : B \to B/J$ and $p_{A(B)} : A \bowtie^f J \to A(B)$ are the canonical projection.

Example

If $B = A[\![X]\!]$(resp., $A[X]$) and $J = XI[\![X]\!]$(resp., $XI[X]$), then $A \bowtie^f J = A + XI[\![X]\!]$(resp., $A + XI[X]$).

The Krull dimension of D

- D denotes an integral domain.

Definition

1. A chain
$$P_0 \subset P_1 \subset \cdots \subset P_n$$
of prime ideals of D is said to have *length* n.
2. For an infinite set Γ, a chain
$$\{P_i\}_{i \in \Gamma}$$
of prime ideals of D is said to have *length* $\mid \Gamma \mid$.

The Krull dimension

The *Krull dimension* of D, denoted by $\dim(D)$, is the supremum of lengths of chains of prime ideals of D.

Valuation rings

Definition

Let G be a totally ordered abelian group, and let K be a field. A valuation of K with values in G is a onto mapping $v : K\backslash\{0\} \to G$ satisfying the following properties

1. $v(xy) = v(x) + v(y)$, for all $x, y \in K\backslash\{0\}$.
2. $v(x + y) \geq \min(v(x), v(y))$, for all $x, y \in K\backslash\{0\}$.

The set $\{x \in K\backslash\{0\} \mid v(x) \geq 0\} \cup \{0\}$ is called a *valuation ring* with value group G.

Definition

A valuation ring V is said to be *discrete* if each primary ideal of V is a power of its radical. In addition, if a valuation ring V is called a *non-discrete* if V is not discrete.

In this talk, denote V as a valuation ring.

SFT-domains

Definition

An integral domain D is an *SFT-domain* (strong finite type domain) if for each ideal I of D, there exist a finitely generated ideal $J \subseteq I$ of D and a positive integer n such that $a^n \in J$ for all $a \in I$.

- Every Noetherian domain is an SFT-domain.
- A finite-dimensional valuation domain is an SFT-domain if and only if it is discrete.
- An SFT valuation domain is a discrete valuation domain, but not vice versa.
- A class of non-SFT-domains contains non-Noetherian almost Dedekind domains and finite-dimensional non-discrete valuation domains.

Ultrafilter

Definition

Let X be a set and $\mathfrak{U} \subset P(X)$. If $\mathfrak{U}$ satisfies the following properties, then $\mathfrak{U}$ is called an *ultrafilter*.

1. The empty set is not an element of $\mathfrak{U}$.
2. If A and B are subsets of X such that $A \subseteq B$ and $A \in \mathfrak{U}$, then $B \in \mathfrak{U}$.
3. If $A, B \in \mathfrak{U}$, then $A \cap B \in \mathfrak{U}$.
4. If A is a subset of X, then either A or $X \backslash A$ is an element of $\mathfrak{U}$.

In particular, if every element in $\mathfrak{U}$ is an infinite set, then $\mathfrak{U}$ is called *non-principal*.

Prime chain ordering

Definition(2011, Luper - Lucas)

A *prime chain ordering* is a pair of nontrivial relations $(\sim, \ll)$ on an integral domain D where $\ll$ is transitive and $\sim$ is an equivalence relation satisfying the followings.

1. For each pair $f, g \in D$, exactly one of $f \ll g, g \ll f$ and $f \sim g$ holds. Especially, $1 \ll 0$.
2. $f \sim uf$ for each unit $u \in D$.
3. If $g \ll f$, then $fg \sim f$ and $f + g \sim g$.
4. If $g \ll f$ and $h \ll f$, then $gh \ll f$.
5. If $f \sim g$, then $fg \sim f$ and either $f + g \sim f$ or $f \ll f + g$.
6. If $e \sim f, g \ll f$ and $g \sim h$, then $h \ll f, g \ll e$ and $h \ll e$.

For a given prime chain $\{P_i\}$, we can define $g \ll f$ if there is prime $P_\beta \in \{P_i\}$ such that $f \in P_\beta$ and $g \notin P_\beta$, and $f \sim g$ if for each $P_\alpha \in \{P_i\}$ either $f, g \in P_\alpha$ or $f, g \notin P_\alpha$. Then the pair $(\sim, \ll)$ is a prime chain ordering.

Prime chain ordering(continued)

Theorem(2011, Luper - Locas)

Let $(\sim, \ll)$ be a prime chain ordering on an integral domain D and let $W = \{d \in D \mid d \ll 0\}$.

1. For each nonempty subset S of W, the set $P_S = \{d \in D \,|\, s \ll d \textit{ for each } s \in S\}$ is a prime ideal of D. Moreover, $P_S = \{d \in D \,|\, s \ll d \textit{ for each } s \in D \backslash P_S\}$.
2. Let $\mathfrak{P}$ be the set of primes of the form P_S. Then $\mathfrak{P}$ is a chain such that $\sim$ is $\sim_{\mathfrak{P}}$ and $\ll$ is $\ll_{\mathfrak{P}}$. Moreover, $\mathfrak{P}$ is closed to unions and intersections.

A phi function

Theorem

Let V be a a one-dimensional nondiscrete valuation domain. Then I is a non-SFT ideal if and only if I is the maximal ideal of V.

To make a prime chain ordering, we define a *phi function*. Let V be a one-dimensional nondiscrete valuation domain and I be a non-SFT ideal of V.

Definition

Let $f(X) = \sum_{n=0}^{\infty} a_n X^n \in V + XI[\![X]\!]$ and $x \in \mathbb{R}^+$,

$$\phi_f(x) = min\{v(a_i) + ix \mid i \geq 0\}$$

.

This function is called a phi function of f.

Prime chain ordering on $V + XI[\![X]\!]$

Relations

Let $\mathfrak{D} = \{d_n\}$ be a strictly decreasing sequence of positive real numbers such that $\lim_{n\to\infty} d_n = 0$ and let $\mathfrak{U}$ be a nonprincipal ultrafilter on $\mathfrak{D}$.

1. $f \sim_{\mathfrak{U}} g$ if there are a positive integer m and a set $U_m \in \mathfrak{U}$ such that $\phi_g(u) \leq m\phi_f(u)$ and $\phi_f(u) \leq m\phi_g(u)$ for each $u \in U_m$.
2. $g \ll_{\mathfrak{U}} f$ if for each positive integer m, there is a set $U_m \in \mathfrak{U}$ such that $m\phi_g(u) < \phi_f(u)$ for each $u \in U_m$.

Theorem

Let V be a non-discrete rank one valuation ring, I be a non-SFT ideal of V and $V + XI[\![X]\!]$ be a composite power series ring. Then the pair $(\sim, \ll)$ is a prime chain ordering on $V + XI[\![X]\!]$.

Convex set(2013, Luper - Locas)

Let $\mathcal{C}$ be the set of nonempty convex subsets of $\mathbb{R}^2$ such that each $A \in \mathcal{C}$ satisfies all of the following properties.

1. For each $(x, y) \in A, 0 \leq x \text{ and } 0 \leq y$.
2. There is the smallest integer $n_0 \geq 0$ such that $(n_0, y) \in A$ for some $y \geq 0$.
3. If $(x, y) \in A$, then $n_0 \leq x, (x, z) \in A$ for all $y \leq z$ and there is the smallest number $w \geq 0$ such that $(x, w) \in A$. Also, there is a point $(x, y) \in A$ with $n_0 < x$.
4. The lower boundary of A is piecewise linear with each corner point of the form (n_m, α_m) with n_m an integer (and $\alpha_m \geq 0$).
5. If there is an $n_0 < x$ such that no point of A has the form (x, y), then there is the largest nonnegative integer n_k and $\alpha_k \geq 0$ such that $(n_k, \alpha_k) \in A$ and $(w, z) \in A$ implies $n_0 \leq w \leq n_k$ with $\alpha_k \leq z$ if $w = \alpha_k$.

Relation between some continuous functions and power series

Let $\mathfrak{C}$ denote the set of continuous functions $h : (0, 1] \to \mathbb{R}^+$ satisfying the following conditions:

(1) $\lim_{x\to 0^+} h(x) = 0^+$.
(2) h'^-, h'^+ exists for each $x > 0$.
(3) $h'^+(x) \geq h'^-(y) \geq h'^+(y) \geq h'^-(z)$ for each $0 < x < y < z$.
(4) there exists a positive integer m such that $mh'^+(x) \geq h'^-(x)$ for each $x > 0$.
(5) $\lim_{x\to 0^+} h'(x) = \infty$.
(6) the set $\{\delta \,|\, h'^-(\delta) \neq h'^+(\delta)\}$ is countable and 0 is the limit point of the set.
(7) $h''^-(x) \leq 0$ and $h''^+(x) \leq 0$ exist for each $x > 0$.

Theorem

Let $h \in \mathfrak{C}$. Then there is a power series f in the set $\{g \in (V + XI[\![X]\!]) \setminus ((U(V) + XI[\![X]\!]) \cup M) \,|\, \lim_{x\to 0^+} \phi_g(x) = 0\}$ such that $h \sim \phi_f$.

Construction

Theorem

Let $\{f_n\}$ be a strictly ascending sequence and $\{g_n\}$ be a strictly descending sequence in $\mathfrak{C}$ such that $f_n \ll g_n$ for each $n \in \mathbb{N}$. Then there exists a function $h \in \mathfrak{C}$ such that $f_n \ll h \ll g_n$ for each $n \in \mathbb{N}$.

$S' = \{f \in (V + XI[\![X]\!]) \mid \lim_{x \to 0^+} \phi_f(x) = 0\}$ and $[S']$ the set of equivalence classes of power series in S' under the relation $\sim$. Consider the set S'_0 consisting of all representatives in $[S']$. Then $[f] \neq [g]$ for distinct $f, g \in S'_0$; $\{[f] \mid f \in S'_0\} = [S']$. Hence $(S'_0, \ll)$ is a totally ordered set.

Definition

Let $(A, \ll)$ be a totally ordered set and B, C be subsets of A. We say $B \ll C$ if $b \ll c$ for each $b \in B$ and $c \in C$. A totally ordered set $(A, \ll)$ is called an *η_1-set* if for any two countable subsets B, C such that $B \ll C$, there exists an element $a \in A$ such that $B \ll \{a\} \ll C$.

η_1-set

Theorem

Let V be a one-dimensional nondiscrete valuation domain, I be a non-SFT ideal of V and $V + XI[\![X]\!]$ a composite power series ring. Then the followings assertions hold.

1. *For $f, g \in V + XI[\![X]\!]$, $P_f \subsetneq P_g$ if and only if $g \ll f$.*
2. *$\{P_f \mid f \in S'_0\}$ is an η_1-set.*

Theorem(2015, Chang - Kang - Toan)

Let X be a nonempty set and let $\mathbf{B} = \{A_i\}_{i \in \wedge}$ be a nonempty family of subsets of X. If $\mathbf{B}$ is totally ordered (under inclusion), then so is the set $\mathbf{B}^* = \{\bigcup_{i \in I} A_i \mid \emptyset \neq I \subseteq \wedge\}$. Furthermore, if $\mathbf{B}$ contains an η_1-set, then the cardinality of $\mathbf{B}^*$ is at least $2^{\aleph_1}$.

Results

Theorem

If V is a rank n discrete valuation domain and I is a nonzero ideal of V, then

$$\dim(V + XI[\![X]\!]) = n + 1.$$

Theorem

If V is a rank one nondiscrete valuation domain and I is a nonzero ideal of V, then

$$\dim(V + XI[\![X]\!]) \geq 2^{\aleph_1}.$$

Unsolved problems

Question1

If D is a non-SFT ring, then

$$dim(D[\![X]\!]) \geq 2^{\aleph_1}.$$

Question2

If D is a non-SFT ring and for any nonzero ideal I of D, then

$$dim(D + XI[\![X]\!]) \geq 2^{\aleph_1}.$$

Thank you for your attention!