A necessary condition for two commutative Noetherian rings to be singularly equivalent

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Introduction

Definition

R: a (not necessarily) commutative Noetherian ring. The singularity category of R is

$$\mathrm{D_{sg}}(R) := \mathrm{D^b}(\mathrm{mod}\, R)/\mathrm{K^b}(\mathrm{proj}\, R).$$

- A triangulated category
- Introduced by Buchweitz by the name of stable derived category
- Measures singularity of R: R is regular $\Leftrightarrow \mathrm{D_{sg}}(R) \cong 0$

Introduction

Definition

Commutative Noetherian rings R, S are sigularly equivalent, denoted by $R \overset{\text{sg}}{\approx} S$, if there is a triangle equivalence $\mathsf{D}_{\mathsf{sg}}(R) \cong \mathsf{D}_{\mathsf{sg}}(S)$.

Example

1. $R \cong S \Rightarrow R \overset{\text{sg}}{\approx} S$
2. R, S: regular $\Rightarrow R \overset{\text{sg}}{\approx} S$
3. (Knörrer's periodicity)
   Let k be an algebraically closed field of characteristic 0 and $0 \neq f \in (x_0, \ldots, x_d) \subseteq k[[x_0, \ldots, x_d]]$. Set $R := k[[x_0, x_1, \ldots, x_d]]/(f)$ and $S := k[[x_0, x_1, \ldots, x_d, u, v]]/(f + uv)$. Then
   $$R \overset{\text{sg}}{\approx} S.$$

Introduction

Observation: all of these singular equivalences, singular loci $\operatorname{Sing} R$ and $\operatorname{Sing} S$ are homeomorphic.

$$\begin{aligned}\operatorname{Sing} S &= \mathrm{V}(f_{x_0}, \dots f_{x_d}, u, v)\\ &\cong \operatorname{Spec}(S/(f_{x_0}, \dots f_{x_d}, u, v))\\ &\cong \operatorname{Spec}(R/(f_{x_0}, \dots f_{x_d}))\\ &\cong \mathrm{V}(f_{x_0}, \dots f_{x_d}) = \operatorname{Sing} R\end{aligned}$$

The first and the last equalities are known as the Jacobian criterion.

Question

Does $R \overset{\text{sg}}{\sim} S \Rightarrow \operatorname{Sing} R \cong \operatorname{Sing} S$ hold?

Notations

Let $\mathcal{T}$ be an essentially small triangulated category and X a topological space.

Definition

1. An additive full subcategory $\mathcal{X}$ of $\mathcal{T}$ is thick if
   - $\mathcal{X}[1] = \mathcal{X}$,
   - for a triangle $L \to M \to N \to L[1]$, $L, N \in \mathcal{X} \Rightarrow M \in \mathcal{X}$,
   - $L \oplus M \in \mathcal{X} \Rightarrow L, M \in \mathcal{T}$.

   $\mathrm{Th}(\mathcal{T}) :=$ { thick subcategories of $\mathcal{T}$}
2. A subset W of X is specialization closed if

$$x \in W \Rightarrow \overline{\{x\}} \subseteq W$$

   ($\Leftrightarrow$ W is a union of closed subsets).
   $\mathrm{Spcl}(X) :=$ { specialization closed subsets of X}

Support theory

$\mathcal{T}$: an essentially small triangulated category

Definition

A support data for $\mathcal{T}$ is a pair (X, σ):

- X: a topological space.
- $\sigma(M)$: a closed subset of X for each $M \in \mathcal{T}$.

such that

1. $\sigma(M) = \emptyset \Leftrightarrow M \cong 0$,
2. $\sigma(M[n]) = \sigma(M)$ for $\forall n \in \mathbb{Z}$,
3. $\sigma(M \oplus N) = \sigma(M) \cup \sigma(N)$,
4. For a triangle $L \to M \to N \to L[1]$, $\sigma(M) \subseteq \sigma(L) \cup \sigma(N)$.

Support theory

Example

1. Let X be a Noetherian scheme. Then the cohomological support:

$$\mathrm{Supp}_X(M) := \{x \in X \mid M_x \not\cong 0 \text{ in } \mathrm{D}^{\mathrm{perf}}(\mathcal{O}_{X,x})\}$$

   defines a support data (X, Supp_X) for the perfect derived category $\mathrm{D}^{\mathrm{perf}}(X)$.

2. Let k be a field and G a finite group. Then support variety gives a support data $(\mathrm{Proj}\,\mathrm{H}^*(G;k), V_G)$ for the stable module category $\underline{\mathrm{mod}}\, kG$.

3. For a commutative Noetherian ring R, the singular support:

$$\mathrm{SSupp}_R(M) := \{\mathfrak{p} \in \mathrm{Sing}\, R \mid M_{\mathfrak{p}} \not\cong 0 \text{ in } \mathrm{D}_{\mathrm{sg}}(R_{\mathfrak{p}})\}$$

   defines a support data $(\mathrm{Sing}\, R, \mathrm{SSupp}_R)$ for $\mathrm{D}_{\mathrm{sg}}(R)$

Support theory

Let (X, σ) be a support data for $\mathcal{T}$.
For $\mathcal{X} \in \mathsf{Th}(\mathcal{T})$ and $W \in \mathsf{Spcl}(X)$,

- $f_\sigma(\mathcal{X}) := \bigcup_{M \in \mathcal{X}} \sigma(M) \in \mathsf{Spcl}(X)$,
- $g_\sigma(W) := \{M \in \mathcal{T} \mid \sigma(M) \subseteq W\} \in \mathsf{Th}(\mathcal{T})$.

Thus, we obtain maps

$$\mathsf{Th}(\mathcal{T}) \underset{g_\sigma}{\overset{f_\sigma}{\rightleftarrows}} \mathsf{Spcl}(X).$$

Support theory

Definition

A support data (X, σ) is called a classifying support data if

1. X is a Noetherian sober space.
2. $f_\sigma \circ g_\sigma = 1$ and $g_\sigma \circ f_\sigma = 1$.

X, $\operatorname{Proj} \mathrm{H}^*(G; k)$, $\operatorname{Sing} R$ are Noetherian and sober.

Theorem A

Let (X, σ) and (X', σ') be classifying support data for essentially small triangulated categories $\mathcal{T}$ and $\mathcal{T}'$, respectively. Then

$$\mathcal{T} \cong \mathcal{T}' \Longrightarrow \exists \varphi : X \xrightarrow{\cong} X' \text{ s.t. } \sigma' = \varphi \circ \sigma$$

holds.

Applications

Theorem [Thomason (1997)]

Let X be a Noetherian scheme and $\mathcal{L}$ an ample line bundle. Then there is a one-to-one correspondence

$$\{\mathcal{X} \in \mathrm{Th}(\mathrm{D}^{\mathrm{perf}}(X)) \mid \mathcal{L}^{-1} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{X} \subseteq \mathcal{X}\} \underset{g_{\mathrm{Supp}}}{\overset{f_{\mathrm{Supp}}}{\rightleftarrows}} \mathrm{Spcl}(X).$$

Corollary

Let X and Y be Noetherian quasi-affine schemes (i.e., open subschemes of affine schemes). Then

$$\begin{aligned} \mathrm{D}^{\mathrm{perf}}(X) \cong \mathrm{D}^{\mathrm{perf}}(Y) &\Longrightarrow X \cong Y\text{: homeo} \\ &\Longrightarrow \dim X = \dim Y. \end{aligned}$$

X is quasi-affine $\Leftrightarrow \mathcal{O}_X$ is ample

Applications

Theorem [Benson-Iyenger-Krause (2012)]

Let k be a field of characteristic p and G be a finite group with $p \mid |G|$. Then there is a one-to-one correspondence

$$\{\mathcal{X} \in \mathrm{Th}(\underline{\mathrm{mod}}\, kG) \mid S \otimes_k \mathcal{X} \subseteq \mathcal{X},\ \forall S : \text{simple}\} \underset{g_{V_G}}{\overset{f_{V_G}}{\rightleftarrows}} \mathrm{Spcl}(\mathrm{Proj}\, \mathrm{H}^*(G;k)).$$

Corollary

Let k (resp. l) be a field of characteristic p (resp. q) and G (resp. H) be a finite p-group (resp. q-group). Then

$$\underline{\mathrm{mod}}\, kG \cong \underline{\mathrm{mod}}\, lH \Longrightarrow \mathrm{Proj}\, \mathrm{H}^*(G;k) \cong \mathrm{Proj}\, \mathrm{H}^*(H;l):\ \text{homeo}$$
$$\Longrightarrow r_p(G) = r_q(H).$$

- G is a p-group $\Rightarrow$ kG has only one simple k
- $\dim \mathrm{H}^*(G;k) = r_p(G) := \sup\{n \mid (\mathbb{Z}/(p))^n \leq G\}$: the p-rank of G.

Singular equivalence

For a commutative Noetherian local ring R, consider the condition:

$$(*) \quad R_{\mathfrak{p}} \text{ is hypersurface for any non-maximal } \mathfrak{p}.$$

Theorem [Takahashi (2010)]

Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring satisfying $(*)$. Then there is a one-to-one correspondence

$$\{\mathcal{X} \in \mathrm{Th}(\mathrm{D_{sg}}(R)) \mid k \in \mathcal{X}\} \underset{g_{\mathsf{SSupp}}}{\overset{f_{\mathsf{SSupp}}}{\rightleftarrows}} \{W \in \mathrm{Spcl}(\mathrm{Sing}\, R) \mid W \neq \emptyset\}.$$

$$\Longrightarrow \mathrm{Th}(\mathrm{D_{sg}}(R)/\operatorname{thick} k) \overset{\cong}{\rightleftarrows} \mathrm{Spcl}(\mathrm{Sing}\, R \setminus \{\mathfrak{m}\}).$$

Problem

Whether the condition "containing the residue field k" is preserved by singular equivalence.

Test objects

Definition

An object $M \in \mathcal{T}$ is a test object if for any $N \in \mathcal{T}$,

$$\mathrm{Hom}_{\mathcal{T}}(M, N[i]) = 0 \text{ for } i \gg 0 \Rightarrow N \cong 0.$$

Remark

1. For a Gorenstein local ring $(R, \mathfrak{m}, k)$, test objects of $\mathrm{D_{sg}}(R) \approx$ test modules T: for any N,

$$\mathrm{Tor}_{\gg 0}(T, N) = 0 \Longrightarrow \mathrm{pd}_R N < \infty.$$

   In particular, k is a test object.

2. Test objects are preserved by triangle equivalences.

Main Theorem

Proposition

Let $(R, \mathfrak{m}, k)$ be a complete intersection ring and T a test object of $\mathrm{D_{sg}}(R)$. Then $k \in \mathrm{thick}_{\mathrm{D_{sg}}(R)}(T)$.

$\Longrightarrow$ for $\mathcal{X} \in \mathrm{Th}(\mathrm{D_{sg}}(R))$, $k \in \mathcal{X}$ iff $\mathcal{X}$ contains a test object.

Corollary

Let $(R, \mathfrak{m}, k)$ and $(S, \mathfrak{n}, l)$ be complete intersection rings. Then

$$R \overset{\mathrm{sg}}{\sim} S \Rightarrow \mathrm{D_{sg}}(R)/\operatorname{thick} k \cong \mathrm{D_{sg}}(S)/\operatorname{thick} l.$$

Theorem B

Let R and S be local complete intersection rings satisfying $(*)$. Then

$$R \overset{\mathrm{sg}}{\sim} S \Longrightarrow \operatorname{Sing} R \cong \operatorname{Sing} S.$$

Applications

Lemma

Let R be a Gorenstein local ring and $\mathfrak{p} \in \operatorname{Spec} R$. Then $\mathcal{X}_\mathfrak{p} := \{M \in \mathrm{D_{sg}}(R) \mid M_\mathfrak{p} \ncong 0\}$ is thick and

$$\mathrm{D_{sg}}(R)/\mathcal{X}_\mathfrak{p} \cong \mathrm{D_{sg}}(R_\mathfrak{p}).$$

Corollary

Let R and S be local complete intersection rings satisfying $(*)$. If $R \overset{\text{sg}}{\sim} S$, then $\exists \varphi : \operatorname{Sing} R \xrightarrow{\cong} \operatorname{Sing} S$, such that

$$R_\mathfrak{p} \overset{\text{sg}}{\sim} S_{\varphi(\mathfrak{p})} \text{ for } \forall \mathfrak{p} \in \operatorname{Sing} R.$$

Applications

Recall (Knörrer's periodicity)

Let k be an algebraically closed field of characteristic 0 and $0 \neq f \in (x_0, \dots x_d) \subseteq k[[x_0, \dots, x_d]]$. Then

$$k[[x_0, \dots, x_d]]/(f) \overset{\text{sg}}{\sim} k[[x_0, \dots, x_d, u, v]]/(f + uv).$$

Corollary

Let k and f be as above. Assume that $k[[x_0, \dots, x_d]]/(f)$ has an isolated singularity. Then

$$k[[x_0, \dots, x_d, y]]/(y^2, f) \overset{\text{sg}}{\nsim} k[[x_0, \dots, x_d, y, u, v]]/(y^2, f + uv).$$

$\Rightarrow$ Knörrer's periodicity fails for non-regular ring $k[[x_0, \dots, x_d, y]]/(y^2)$.

Thank you for your kind attention.