The moduli of subalgebras of the full matrix ring of degree 3

Kazunori Nakamoto
Takeshi Torii

October 7, 2017

Definition 1

Let k be an algebraically closed field.

Introduction

Definition 1

Let k be an algebraically closed field. Let A, B be k-subalgebras of $\mathrm{M}_n(k)$.

Introduction

Definition 1

Let k be an algebraically closed field. Let A, B be k-subalgebras of $\mathrm{M}_n(k)$. We say that A and B are *equivalent* if there exists $P \in \mathrm{GL}_n(k)$ such that $P^{-1}AP = B$.

Theorem 2

There exist 26 *equivalence classes of* k*-subalgebras of* $\mathrm{M}_3(k)$ *for any algebraically closed field* k*.*

Theorem 2

There exist 26 equivalence classes of k-subalgebras of $\mathrm{M}_3(k)$ for any algebraically closed field k.

(1) $\mathrm{M}_3(k)$

(2) $\mathrm{P}_{2,1}(k) := \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \in \mathrm{M}_3(k) \right\}$

(3) $\mathrm{P}_{1,2}(k) := \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in \mathrm{M}_3(k) \right\}$

(4) $\mathrm{B}_3(k) := \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in \mathrm{M}_3(k) \right\}$

(5) $\mathrm{C}_3(k) := \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \middle| \ a \in k \right\}$

$$\text{(6)}\quad \mathrm{D}_3(k) := \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \in \mathrm{M}_3(k) \right\}$$

$$\text{(7)}\quad (\mathrm{C}_2 \times \mathrm{D}_1)(k) := \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \middle| \, a, b \in k \right\}$$

$$\text{(8)}\quad (\mathrm{N}_2 \times \mathrm{D}_1)(k) := \left\{ \begin{pmatrix} a & c & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \middle| \, a, b, c \in k \right\}$$

$$\text{(9)}\quad (\mathrm{B}_2 \times \mathrm{D}_1)(k) := \left\{ \begin{pmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \in \mathrm{M}_3(k) \right\}$$

$$\text{(10)}\quad (\mathrm{M}_2 \times \mathrm{D}_1)(k) := \left\{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \in \mathrm{M}_3(k) \right\}$$

$$(11)\quad \mathrm{J}_3(k) := \left\{ \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} \middle| \, a, b, c \in k \right\}$$

$$(12)\quad \mathrm{N}_3(k) := \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \middle| \, a, b, c, d \in k \right\}$$

$$(13)\quad S_1(k) := \left\{ \begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \middle| \, a, b \in k \right\}$$

$$(14)\quad S_2(k) := \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & c \\ 0 & 0 & b \end{pmatrix} \middle| \, a, b, c \in k \right\}$$

$$(15)\quad S_3(k) := \left\{ \begin{pmatrix} a & 0 & c \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix} \middle| \, a, b, c \in k \right\}$$

$$(16)\quad S_4(k) := \left\{ \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \middle| \, a, b, c \in k \right\}$$

(17) $$S_5(k) := \left\{ \begin{pmatrix} a & 0 & b \\ 0 & a & c \\ 0 & 0 & a \end{pmatrix} \middle| \ a, b, c \in k \right\}$$

(18) $$S_6(k) := \left\{ \begin{pmatrix} a & c & d \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \middle| \ a, b, c, d \in k \right\}$$

(19) $$S_7(k) := \left\{ \begin{pmatrix} a & 0 & c \\ 0 & a & d \\ 0 & 0 & b \end{pmatrix} \middle| \ a, b, c, d \in k \right\}$$

(20) $$S_8(k) := \left\{ \begin{pmatrix} a & c & d \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix} \middle| \ a, b, c, d \in k \right\}$$

(21) $$S_9(k) := \left\{ \begin{pmatrix} a & 0 & c \\ 0 & b & d \\ 0 & 0 & b \end{pmatrix} \middle| \ a, b, c, d \in k \right\}$$

$$(22)\quad S_{10}(k) := \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & e \end{pmatrix} \middle| \, a, b, c, d, e \in k \right\}$$

$$(23)\quad S_{11}(k) := \left\{ \begin{pmatrix} a & b & c \\ 0 & e & d \\ 0 & 0 & a \end{pmatrix} \middle| \, a, b, c, d, e \in k \right\}$$

$$(24)\quad S_{12}(k) := \left\{ \begin{pmatrix} a & b & c \\ 0 & e & d \\ 0 & 0 & e \end{pmatrix} \middle| \, a, b, c, d, e \in k \right\}$$

$$(25)\quad S_{13}(k) := \left\{ \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \in \mathrm{M}_3(k) \right\}$$

$$(26)\quad S_{14}(k) := \left\{ \begin{pmatrix} * & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in \mathrm{M}_3(k) \right\}$$

Today's talk

- Let us consider the moduli $\mathrm{Mold}_{3,d}$ of d-dimensional subalgebras of M_3.

Today's talk

- Let us consider the moduli $\mathrm{Mold}_{3,d}$ of d-dimensional subalgebras of M_3.
- $\mathrm{Mold}_{3,d}$ is a closed subscheme of the Grassmann scheme $\mathrm{Grass}(d, 9)$.

Today's talk

- Let us consider the moduli $\mathrm{Mold}_{3,d}$ of d-dimensional subalgebras of M_3.
- $\mathrm{Mold}_{3,d}$ is a closed subscheme of the Grassmann scheme $\mathrm{Grass}(d, 9)$.
- We talk about the cases $d = 2$ and $d = 3$.

For introducing the moduli of subalgebras of the full matrix ring, we need to define molds on schemes.

For introducing the moduli of subalgebras of the full matrix ring, we need to define molds on schemes.

Definition 3

Let X be a scheme. A subsheaf of $\mathcal{O}_X$-algebras $\mathcal{A} \subseteq \mathrm{M}_n(\mathcal{O}_X)$ is said to be a *mold* of degree n on X if $\mathcal{A}$ and $\mathrm{M}_n(\mathcal{O}_X)/\mathcal{A}$ are locally free sheaves on X.

For introducing the moduli of subalgebras of the full matrix ring, we need to define molds on schemes.

Definition 3

Let X be a scheme. A subsheaf of $\mathcal{O}_X$-algebras $\mathcal{A} \subseteq \mathrm{M}_n(\mathcal{O}_X)$ is said to be a *mold* of degree n on X if $\mathcal{A}$ and $\mathrm{M}_n(\mathcal{O}_X)/\mathcal{A}$ are locally free sheaves on X. We denote by $\mathrm{rank}\mathcal{A}$ the rank of $\mathcal{A}$ as a locally free sheaf on X. For a commutative ring R, we say that an R-subalgebra $A \subseteq \mathrm{M}_n(R)$ is a *mold* of degree n over R if A is a mold of degree n on $\mathrm{Spec}R$.

Proposition 1.1

The following contravariant functor is representable by a $\mathbb{Z}$-scheme $\mathrm{Mold}_{n,d}$.

$$\begin{array}{cccl} \mathrm{Mold}_{n,d} & : & (\mathbf{Sch})^{op} & \to & (\mathbf{Sets}) \\ & & X & \mapsto & \left\{ \mathcal{A} \,\middle|\, \mathcal{A} : \text{rank } d \text{ mold of degree } n \text{ on } X \right\} \end{array}$$

Proposition 1.1

The following contravariant functor is representable by a $\mathbb{Z}$-scheme $\mathrm{Mold}_{n,d}$.

$$\begin{array}{ccccc} \mathrm{Mold}_{n,d} & : & (\mathbf{Sch})^{op} & \to & (\mathbf{Sets}) \\ & & X & \mapsto & \left\{ \mathcal{A} \,\middle|\, \mathcal{A} : \textit{rank } d \textit{ mold of degree } n \textit{ on } X \right\} \end{array}$$

Moreover, $\mathrm{Mold}_{n,d}$ is a closed subscheme of the Grassmann scheme $\mathrm{Grass}(d, n^2)$.

Example 4

Let $n = 3$. If $d = 1$ or $d \geq 6$, then

$$\begin{aligned} \mathrm{Mold}_{3,1} &= \mathrm{Spec}\mathbb{Z}, \\ \mathrm{Mold}_{3,6} &= \mathrm{Flag} := \mathrm{GL}_3/\{(a_{ij}) \in \mathrm{GL}_3 \mid a_{ij} = 0 \text{ for } i > j\}, \\ \mathrm{Mold}_{3,7} &= \mathbb{P}^2_{\mathbb{Z}} \coprod \mathbb{P}^2_{\mathbb{Z}}, \\ \mathrm{Mold}_{3,8} &= \emptyset, \\ \mathrm{Mold}_{3,9} &= \mathrm{Spec}\mathbb{Z}. \end{aligned}$$

Example 4

Let $n = 3$. If $d = 1$ or $d \geq 6$, then

$$\begin{aligned}
\mathrm{Mold}_{3,1} &= \mathrm{Spec}\mathbb{Z}, \\
\mathrm{Mold}_{3,6} &= \mathrm{Flag} := \mathrm{GL}_3/\{(a_{ij}) \in \mathrm{GL}_3 \mid a_{ij} = 0 \text{ for } i > j\}, \\
\mathrm{Mold}_{3,7} &= \mathbb{P}^2_{\mathbb{Z}} \coprod \mathbb{P}^2_{\mathbb{Z}}, \\
\mathrm{Mold}_{3,8} &= \emptyset, \\
\mathrm{Mold}_{3,9} &= \mathrm{Spec}\mathbb{Z}.
\end{aligned}$$

When $d = 1$, $A = RI_3 \subset \mathrm{M}_3(R)$ corresponds to the unique R-point of $\mathrm{Mold}_{3,1} = \mathrm{Spec}\mathbb{Z}$ for a commutative ring R.

Example 4

Let $n = 3$. If $d = 1$ or $d \geq 6$, then

$$\begin{aligned}
\mathrm{Mold}_{3,1} &= \mathrm{Spec}\mathbb{Z}, \\
\mathrm{Mold}_{3,6} &= \mathrm{Flag} := \mathrm{GL}_3/\{(a_{ij}) \in \mathrm{GL}_3 \mid a_{ij} = 0 \text{ for } i > j\}, \\
\mathrm{Mold}_{3,7} &= \mathbb{P}^2_{\mathbb{Z}} \coprod \mathbb{P}^2_{\mathbb{Z}}, \\
\mathrm{Mold}_{3,8} &= \emptyset, \\
\mathrm{Mold}_{3,9} &= \mathrm{Spec}\mathbb{Z}.
\end{aligned}$$

When $d = 1$, $A = RI_3 \subset \mathrm{M}_3(R)$ corresponds to the unique R-point of $\mathrm{Mold}_{3,1} = \mathrm{Spec}\mathbb{Z}$ for a commutative ring R.
When $d = 9$, $A = \mathrm{M}_3(R)$ corresponds to the unique R-point of $\mathrm{Mold}_{3,9} = \mathrm{Spec}\mathbb{Z}$ for a commutative ring R.

Example 4

Let $n = 3$. If $d = 1$ or $d \geq 6$, then

$$\begin{aligned}
\mathrm{Mold}_{3,1} &= \mathrm{Spec}\mathbb{Z}, \\
\mathrm{Mold}_{3,6} &= \mathrm{Flag} := \mathrm{GL}_3/\{(a_{ij}) \in \mathrm{GL}_3 \mid a_{ij} = 0 \text{ for } i > j\}, \\
\mathrm{Mold}_{3,7} &= \mathbb{P}^2_{\mathbb{Z}} \coprod \mathbb{P}^2_{\mathbb{Z}}, \\
\mathrm{Mold}_{3,8} &= \emptyset, \\
\mathrm{Mold}_{3,9} &= \mathrm{Spec}\mathbb{Z}.
\end{aligned}$$

When $d = 1$, $A = RI_3 \subset \mathrm{M}_3(R)$ corresponds to the unique R-point of $\mathrm{Mold}_{3,1} = \mathrm{Spec}\mathbb{Z}$ for a commutative ring R.
When $d = 9$, $A = \mathrm{M}_3(R)$ corresponds to the unique R-point of $\mathrm{Mold}_{3,9} = \mathrm{Spec}\mathbb{Z}$ for a commutative ring R.

When $d = 6$, the set of k-rational points of $\mathrm{Mold}_{3,6} = \mathrm{Flag}$ coincides with $\{P\mathrm{B}_3(k)P^{-1} \mid P \in \mathrm{GL}_3(k)\}$ for a field k, where

$$\mathrm{B}_3(k) := \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in \mathrm{M}_3(k) \right\}.$$

When $d = 6$, the set of k-rational points of $\mathrm{Mold}_{3,6} = \mathrm{Flag}$ coincides with $\{P\mathrm{B}_3(k)P^{-1} \mid P \in \mathrm{GL}_3(k)\}$ for a field k, where

$$\mathrm{B}_3(k) := \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in \mathrm{M}_3(k) \right\}.$$

When $d = 7$, $\mathrm{Mold}_{3,7} = \mathbb{P}^2_{\mathbb{Z}} \coprod \mathbb{P}^2_{\mathbb{Z}}$.

When $d = 6$, the set of k-rational points of $\mathrm{Mold}_{3,6} = \mathrm{Flag}$ coincides with $\{P\mathrm{B}_3(k)P^{-1} \mid P \in \mathrm{GL}_3(k)\}$ for a field k, where

$$\mathrm{B}_3(k) := \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in \mathrm{M}_3(k) \right\}.$$

When $d = 7$, $\mathrm{Mold}_{3,7} = \mathbb{P}^2_{\mathbb{Z}} \coprod \mathbb{P}^2_{\mathbb{Z}}$.

Let $\mathrm{P}_{2,1}(k) := \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \in \mathrm{M}_3(k) \right\}$ and

$$\mathrm{P}_{1,2}(k) := \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in \mathrm{M}_3(k) \right\}.$$

When $d = 6$, the set of k-rational points of $\mathrm{Mold}_{3,6} = \mathrm{Flag}$ coincides with $\{P\mathrm{B}_3(k)P^{-1} \mid P \in \mathrm{GL}_3(k)\}$ for a field k, where

$$\mathrm{B}_3(k) := \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in \mathrm{M}_3(k) \right\}.$$

When $d = 7$, $\mathrm{Mold}_{3,7} = \mathbb{P}^2_{\mathbb{Z}} \coprod \mathbb{P}^2_{\mathbb{Z}}$.

Let $\mathrm{P}_{2,1}(k) := \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \in \mathrm{M}_3(k) \right\}$ and

$$\mathrm{P}_{1,2}(k) := \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in \mathrm{M}_3(k) \right\}.$$

The set of k-rational points of $\mathrm{Mold}_{3,7} = \mathbb{P}^2_{\mathbb{Z}} \coprod \mathbb{P}^2_{\mathbb{Z}}$ coincides with

$$\{P\mathrm{P}_{2,1}(k)P^{-1} \mid P \in \mathrm{GL}_3(k)\} \coprod \{P\mathrm{P}_{1,2}(k)P^{-1} \mid P \in \mathrm{GL}_3(k)\},$$

where k is a field.

In this talk, we discuss $\mathrm{Mold}_{3,2}$ and $\mathrm{Mold}_{3,3}$.

In this talk, we discuss $\mathrm{Mold}_{3,2}$ and $\mathrm{Mold}_{3,3}$.

First we deal with $\mathrm{Mold}_{3,2}$.

Let k be an algebraically closed field. There exist two equivalence classes of 2-dimensional k-subalgebras of $\mathrm{M}_3(k)$:

$$(\mathrm{C}_2 \times \mathrm{D}_1)(k) := \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \middle| \, a, b \in k \right\}$$

and

$$S_1(k) := \left\{ \begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \middle| \, a, b \in k \right\}.$$

Subregular matrix

We can classify 3×3-matrices into three types: Regular matrices, subregular matrices, and scalar matrices.

Subregular matrix

We can classify 3×3-matrices into three types: Regular matrices, subregular matrices, and scalar matrices.

Definition 5

Let M_3 be the scheme of 3×3-matrices over $\mathbb{Z}$.

Subregular matrix

We can classify 3×3-matrices into three types: Regular matrices, subregular matrices, and scalar matrices.

Definition 5

Let M_3 be the scheme of 3×3-matrices over $\mathbb{Z}$. In other words, $\mathrm{M}_3 \cong \mathbb{A}^9_{\mathbb{Z}}$ and we can consider a universal matrix A on M_3.

Subregular matrix

We can classify 3×3-matrices into three types: Regular matrices, subregular matrices, and scalar matrices.

Definition 5

Let M_3 be the scheme of 3×3-matrices over $\mathbb{Z}$. In other words, $\mathrm{M}_3 \cong \mathbb{A}^9_{\mathbb{Z}}$ and we can consider a universal matrix A on M_3. Then M_3 can be divided into the following three subschemes:

$$\mathrm{M}_3 = \mathrm{M}_3^{\mathrm{reg}} \coprod \mathrm{M}_3^{\mathrm{sr}} \coprod \mathrm{M}_3^{\mathrm{scalar}}.$$

Subregular matrix

We can classify 3×3-matrices into three types: Regular matrices, subregular matrices, and scalar matrices.

Definition 5

Let M_3 be the scheme of 3×3-matrices over $\mathbb{Z}$. In other words, $\mathrm{M}_3 \cong \mathbb{A}^9_{\mathbb{Z}}$ and we can consider a universal matrix A on M_3. Then M_3 can be divided into the following three subschemes:

$$\mathrm{M}_3 = \mathrm{M}_3^{\mathrm{reg}} \coprod \mathrm{M}_3^{\mathrm{sr}} \coprod \mathrm{M}_3^{\mathrm{scalar}}.$$

Here $\mathrm{M}_3^{\mathrm{reg}}$ is an open subscheme consisting of non-derogatory matrices (or regular matrices),

Subregular matrix

We can classify 3×3-matrices into three types: Regular matrices, subregular matrices, and scalar matrices.

Definition 5

Let M_3 be the scheme of 3×3-matrices over $\mathbb{Z}$. In other words, $\mathrm{M}_3 \cong \mathbb{A}^9_{\mathbb{Z}}$ and we can consider a universal matrix A on M_3. Then M_3 can be divided into the following three subschemes:

$$\mathrm{M}_3 = \mathrm{M}_3^{\mathrm{reg}} \coprod \mathrm{M}_3^{\mathrm{sr}} \coprod \mathrm{M}_3^{\mathrm{scalar}}.$$

Here $\mathrm{M}_3^{\mathrm{reg}}$ is an open subscheme consisting of non-derogatory matrices (or regular matrices), $\mathrm{M}_3^{\mathrm{scalar}}$ is a closed subschemes consisting of scalar matrices,

Subregular matrix

We can classify 3×3-matrices into three types: Regular matrices, subregular matrices, and scalar matrices.

Definition 5

Let M_3 be the scheme of 3×3-matrices over $\mathbb{Z}$. In other words, $\mathrm{M}_3 \cong \mathbb{A}^9_{\mathbb{Z}}$ and we can consider a universal matrix A on M_3. Then M_3 can be divided into the following three subschemes:

$$\mathrm{M}_3 = \mathrm{M}_3^{\mathrm{reg}} \coprod \mathrm{M}_3^{\mathrm{sr}} \coprod \mathrm{M}_3^{\mathrm{scalar}}.$$

Here $\mathrm{M}_3^{\mathrm{reg}}$ is an open subscheme consisting of non-derogatory matrices (or regular matrices), $\mathrm{M}_3^{\mathrm{scalar}}$ is a closed subschemes consisting of scalar matrices, and $\mathrm{M}_3^{\mathrm{sr}}$ is a subscheme consisting of matrices A satisfying the conditions that A^2 can be written as a linear combination of I_3 and A and that I_3 and A are linearly independent.

Roughly speaking, if the degree of the minimal polynomial for a 3×3-matrix A is 3, 2, or 1, then we call A regular, subregular, or scalar, respectively.

Roughly speaking, if the degree of the minimal polynomial for a 3×3-matrix A is 3, 2, or 1, then we call A regular, subregular, or scalar, respectively.

For describing $\mathrm{Mold}_{3,2}$, we deal with subregular matrices.

The normal form of subregular matrices

The normal form of subregular matrices

Proposition 1.2

Let R be a local ring. For $A \in \mathrm{M}_3^{\mathrm{sr}}(R)$, there exists $P \in \mathrm{GL}_3(R)$ such that

$$P^{-1}AP = \begin{pmatrix} a & 1 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}.$$

The normal form of subregular matrices

Proposition 1.2

Let R be a local ring. For $A \in \mathrm{M}_3^{\mathrm{sr}}(R)$, there exists $P \in \mathrm{GL}_3(R)$ such that

$$P^{-1}AP = \begin{pmatrix} a & 1 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}.$$

Moreover, $a, b \in R$ are determined by only A (not by P).

The normal form of subregular matrices

Proposition 1.2

Let R be a local ring. For $A \in \mathrm{M}_3^{\mathrm{sr}}(R)$, there exists $P \in \mathrm{GL}_3(R)$ such that

$$P^{-1}AP = \begin{pmatrix} a & 1 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}.$$

Moreover, $a, b \in R$ are determined by only A (not by P).

Definition 6

We call $a, b \in R$ in Proposition 1.2 the *a-invariant* and the *b-invariant* of A, respectively.

The subscheme $\mathrm{M}_3^{\mathrm{sr}}$ is the moduli of 3×3 subregular matrices. For the universal subregular matrix A on $\mathrm{M}_3^{\mathrm{sr}}$, we can define the a-invariant and the b-invariant of A on $\mathrm{M}_3^{\mathrm{sr}}$.

The subscheme $\mathrm{M}_3^{\mathrm{sr}}$ is the moduli of 3×3 subregular matrices. For the universal subregular matrix A on $\mathrm{M}_3^{\mathrm{sr}}$, we can define the a-invariant and the b-invariant of A on $\mathrm{M}_3^{\mathrm{sr}}$.

Definition 7

We denote by $a(A), b(A) \in \mathcal{O}_{\mathrm{M}_3^{\mathrm{sr}}}(\mathrm{M}_3^{\mathrm{sr}})$ the a-invariant and b-invariant of the universal matrix A on $\mathrm{M}_3^{\mathrm{sr}}$, respectively.

The subscheme $\mathrm{M}_3^{\mathrm{sr}}$ is the moduli of 3×3 subregular matrices. For the universal subregular matrix A on $\mathrm{M}_3^{\mathrm{sr}}$, we can define the a-invariant and the b-invariant of A on $\mathrm{M}_3^{\mathrm{sr}}$.

Definition 7

We denote by $a(A), b(A) \in \mathcal{O}_{\mathrm{M}_3^{\mathrm{sr}}}(\mathrm{M}_3^{\mathrm{sr}})$ the a-invariant and b-invariant of the universal matrix A on $\mathrm{M}_3^{\mathrm{sr}}$, respectively. These are PGL_3-invariant, where the group scheme PGL_3 acts on $\mathrm{M}_3^{\mathrm{sr}}$ by $A \mapsto P^{-1}AP$.

The quotient of $\mathrm{M}_3^{\mathrm{sr}}$ by PGL_3 is isomorphic to $\mathbb{A}^2_{\mathbb{Z}}$.

The quotient of $\mathrm{M}_3^{\mathrm{sr}}$ by PGL_3 is isomorphic to $\mathbb{A}_{\mathbb{Z}}^2$.

Proposition 1.3

Let $\pi : \mathrm{M}_3^{\mathrm{sr}} \to \mathbb{A}_{\mathbb{Z}}^2$ *be the morphism defined by* $A \mapsto (a(A), b(A))$.

The quotient of $\mathrm{M}_3^{\mathrm{sr}}$ by PGL_3 is isomorphic to $\mathbb{A}^2_{\mathbb{Z}}$.

Proposition 1.3

Let $\pi : \mathrm{M}_3^{\mathrm{sr}} \to \mathbb{A}^2_{\mathbb{Z}}$ *be the morphism defined by* $A \mapsto (a(A), b(A))$*. Then* π *gives a universal geometric quotient by* PGL_3*.*

The quotient of $\mathrm{M}_3^{\mathrm{sr}}$ by PGL_3 is isomorphic to $\mathbb{A}^2_{\mathbb{Z}}$.

Proposition 1.3

Let $\pi : \mathrm{M}_3^{\mathrm{sr}} \to \mathbb{A}^2_{\mathbb{Z}}$ *be the morphism defined by* $A \mapsto (a(A), b(A))$*. Then* π *gives a universal geometric quotient by* PGL_3*. Moreover,* $\mathrm{M}_3^{\mathrm{sr}}$ *is a smooth integral scheme of relative dimension 6 over* $\mathbb{Z}$*.*

There exists a surjective morphism $\phi : \mathrm{M}_3^{\mathrm{sr}} \to \mathrm{Mold}_{3,2}$.

There exists a surjective morphism $\phi : \mathrm{M}_3^{\mathrm{sr}} \to \mathrm{Mold}_{3,2}$.

The scheme $\mathrm{Mold}_{3,2}$ is the moduli of 2-dimensional subalgebras of M_3.

There exists a surjective morphism $\phi : \mathrm{M}_3^{\mathrm{sr}} \to \mathrm{Mold}_{3,2}$.

The scheme $\mathrm{Mold}_{3,2}$ is the moduli of 2-dimensional subalgebras of M_3.

Let $\langle A \rangle$ be the subalgebra generated by A for $A \in \mathrm{M}_3^{\mathrm{sr}}$. We define $\phi : \mathrm{M}_3^{\mathrm{sr}} \to \mathrm{Mold}_{3,2}$ by $A \mapsto \langle A \rangle$.

There exists a surjective morphism $\phi : \mathrm{M}_3^{\mathrm{sr}} \to \mathrm{Mold}_{3,2}$.

The scheme $\mathrm{Mold}_{3,2}$ is the moduli of 2-dimensional subalgebras of M_3.

Let $\langle A \rangle$ be the subalgebra generated by A for $A \in \mathrm{M}_3^{\mathrm{sr}}$. We define $\phi : \mathrm{M}_3^{\mathrm{sr}} \to \mathrm{Mold}_{3,2}$ by $A \mapsto \langle A \rangle$.

Proposition 1.4

The morphism $\phi : \mathrm{M}_3^{\mathrm{sr}} \to \mathrm{Mold}_{3,2}$ is smooth and surjective.

What is $\mathrm{Mold}_{3,2}$?

What is $\mathrm{Mold}_{3,2}$?

We describe $\mathrm{Mold}_{3,2}$ explicitly.

What is $\mathrm{Mold}_{3,2}$?

We describe $\mathrm{Mold}_{3,2}$ explicitly.

Let $V := \mathcal{O}_{\mathbb{Z}}^{\oplus 3}$ be a rank 3 trivial vector bundle on $\mathrm{Spec}\mathbb{Z}$.

What is $\mathrm{Mold}_{3,2}$?

We describe $\mathrm{Mold}_{3,2}$ explicitly.

Let $V := \mathcal{O}_{\mathbb{Z}}^{\oplus 3}$ be a rank 3 trivial vector bundle on $\mathrm{Spec}\mathbb{Z}$. We denote by $\mathbb{P}_*(V)$ the projective plane consisting of subline bundles of V.

What is $\mathrm{Mold}_{3,2}$?

We describe $\mathrm{Mold}_{3,2}$ explicitly.

Let $V := \mathcal{O}_{\mathbb{Z}}^{\oplus 3}$ be a rank 3 trivial vector bundle on $\mathrm{Spec}\mathbb{Z}$. We denote by $\mathbb{P}_*(V)$ the projective plane consisting of subline bundles of V. We also denote by $\mathbb{P}^*(V)$ the projective plane consisting of rank 2 subbundles of V.

What is $\mathrm{Mold}_{3,2}$?

We describe $\mathrm{Mold}_{3,2}$ explicitly.

Let $V := \mathcal{O}_{\mathbb{Z}}^{\oplus 3}$ be a rank 3 trivial vector bundle on $\mathrm{Spec}\mathbb{Z}$. We denote by $\mathbb{P}_*(V)$ the projective plane consisting of subline bundles of V. We also denote by $\mathbb{P}^*(V)$ the projective plane consisting of rank 2 subbundles of V.

Let us define a morphism $\xi : \mathbb{P}_*(V) \times \mathbb{P}^*(V) \to \mathrm{Mold}_{3,2}$.

What is $\mathrm{Mold}_{3,2}$?

We describe $\mathrm{Mold}_{3,2}$ explicitly.

Let $V := \mathcal{O}_{\mathbb{Z}}^{\oplus 3}$ be a rank 3 trivial vector bundle on $\mathrm{Spec}\mathbb{Z}$. We denote by $\mathbb{P}_*(V)$ the projective plane consisting of subline bundles of V. We also denote by $\mathbb{P}^*(V)$ the projective plane consisting of rank 2 subbundles of V.

Let us define a morphism $\xi : \mathbb{P}_*(V) \times \mathbb{P}^*(V) \to \mathrm{Mold}_{3,2}$.

From now on, we omit scheme-theoretical proofs. In the following discussions, we recognize vector bundles as vector spaces over a field k, for simplicity.

$\xi : \mathbb{P}_*(V) \times \mathbb{P}^*(V) \to \mathrm{Mold}_{3,2}$

$\xi : \mathbb{P}_*(V) \times \mathbb{P}^*(V) \to \mathrm{Mold}_{3,2}$

Let $(L, W) \in \mathbb{P}_*(V) \times \mathbb{P}^*(V)$. In other words, let L and W be a 1-dimensional subspace and a 2-dimensional subspace of V, respectively.

$\xi : \mathbb{P}_*(V) \times \mathbb{P}^*(V) \to \mathrm{Mold}_{3,2}$

Let $(L, W) \in \mathbb{P}_*(V) \times \mathbb{P}^*(V)$. In other words, let L and W be a 1-dimensional subspace and a 2-dimensional subspace of V, respectively. We can regard $f \in \mathrm{Hom}_k(V/W, L)$ as an element of $\mathrm{End}_k(V)$ by

$$V \stackrel{\text{proj.}}{\to} V/W \stackrel{f}{\to} L \hookrightarrow V.$$

$\xi : \mathbb{P}_*(V) \times \mathbb{P}^*(V) \to \mathrm{Mold}_{3,2}$

Let $(L, W) \in \mathbb{P}_*(V) \times \mathbb{P}^*(V)$. In other words, let L and W be a 1-dimensional subspace and a 2-dimensional subspace of V, respectively. We can regard $f \in \mathrm{Hom}_k(V/W, L)$ as an element of $\mathrm{End}_k(V)$ by

$$V \overset{\text{proj.}}{\to} V/W \overset{f}{\to} L \hookrightarrow V.$$

We denote by $\xi(L, W)$ the k-subalgebra of $\mathrm{End}_k(V)$ generated by $\{f \in \mathrm{Hom}_k(V/W, L)\}$ and id_V.

$\xi : \mathbb{P}_*(V) \times \mathbb{P}^*(V) \to \mathrm{Mold}_{3,2}$

Let $(L, W) \in \mathbb{P}_*(V) \times \mathbb{P}^*(V)$. In other words, let L and W be a 1-dimensional subspace and a 2-dimensional subspace of V, respectively. We can regard $f \in \mathrm{Hom}_k(V/W, L)$ as an element of $\mathrm{End}_k(V)$ by

$$V \stackrel{\text{proj.}}{\to} V/W \stackrel{f}{\to} L \hookrightarrow V.$$

We denote by $\xi(L, W)$ the k-subalgebra of $\mathrm{End}_k(V)$ generated by $\{f \in \mathrm{Hom}_k(V/W, L)\}$ and id_V. Since $\dim \mathrm{Hom}_k(V/W, L) = 1$, we see that $\xi(L, W)$ is a 2-dimensional k-subalgebra.

$\xi : \mathbb{P}_*(V) \times \mathbb{P}^*(V) \to \mathrm{Mold}_{3,2}$

Let $(L, W) \in \mathbb{P}_*(V) \times \mathbb{P}^*(V)$. In other words, let L and W be a 1-dimensional subspace and a 2-dimensional subspace of V, respectively. We can regard $f \in \mathrm{Hom}_k(V/W, L)$ as an element of $\mathrm{End}_k(V)$ by

$$V \stackrel{\text{proj.}}{\to} V/W \stackrel{f}{\to} L \hookrightarrow V.$$

We denote by $\xi(L, W)$ the k-subalgebra of $\mathrm{End}_k(V)$ generated by $\{f \in \mathrm{Hom}_k(V/W, L)\}$ and id_V. Since $\dim \mathrm{Hom}_k(V/W, L) = 1$, we see that $\xi(L, W)$ is a 2-dimensional k-subalgebra. We define a morphism $\xi : \mathbb{P}_*(V) \times \mathbb{P}^*(V) \to \mathrm{Mold}_{3,2}$ by $(L, W) \mapsto \xi(L, W)$.

$\mathrm{Mold}_{3,2} \cong \mathbb{P}_*(V) \times \mathbb{P}^*(V)$

Theorem 8

The morphism $\xi : \mathbb{P}_*(V) \times \mathbb{P}^*(V) \to \mathrm{Mold}_{3,2}$ *is an isomorphism.*

Two-dimensional subalgebras $(\mathrm{C}_2 \times \mathrm{D}_1)$ and S_1

Two-dimensional subalgebras $(\mathrm{C}_2 \times \mathrm{D}_1)$ and S_1

There are two types of 2-dimensional k-subalgebras of $\mathrm{M}_3(k)$ over an algebraically closed field k:

Two-dimensional subalgebras $(\mathrm{C}_2 \times \mathrm{D}_1)$ and S_1

There are two types of 2-dimensional k-subalgebras of $\mathrm{M}_3(k)$ over an algebraically closed field k:

$$(\mathrm{C}_2 \times \mathrm{D}_1)(k) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \middle| \, a, b \in k \right\},$$

$$S_1(k) = \left\{ \begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \middle| \, a, b \in k \right\}$$

How $(\mathrm{C}_2 \times \mathrm{D}_1)$ and S_1 are contained in $\mathrm{Mold}_{3,2}$?

How $(\mathrm{C}_2 \times \mathrm{D}_1)$ and S_1 are contained in $\mathrm{Mold}_{3,2}$?

Definition 9

We define an open subscheme $\mathrm{M}_3^{\mathrm{C}_2\times\mathrm{D}_1}$ of $\mathrm{M}_3^{\mathrm{sr}}$ by

$$\mathrm{M}_3^{\mathrm{C}_2\times\mathrm{D}_1} := \{A \in \mathrm{M}_3^{\mathrm{sr}} \mid a(A) - b(A) \neq 0\}.$$

We also define a closed subscheme $\mathrm{M}_3^{S_1}$ of $\mathrm{M}_3^{\mathrm{sr}}$ by

$$\mathrm{M}_3^{S_1} := \{A \in \mathrm{M}_3^{\mathrm{sr}} \mid a(A) - b(A) = 0\}.$$

How $(\mathrm{C}_2 \times \mathrm{D}_1)$ and S_1 are contained in $\mathrm{Mold}_{3,2}$?

Definition 9

We define an open subscheme $\mathrm{M}_3^{\mathrm{C}_2 \times \mathrm{D}_1}$ of $\mathrm{M}_3^{\mathrm{sr}}$ by

$$\mathrm{M}_3^{\mathrm{C}_2 \times \mathrm{D}_1} := \{A \in \mathrm{M}_3^{\mathrm{sr}} \mid a(A) - b(A) \neq 0\}.$$

We also define a closed subscheme $\mathrm{M}_3^{S_1}$ of $\mathrm{M}_3^{\mathrm{sr}}$ by

$$\mathrm{M}_3^{S_1} := \{A \in \mathrm{M}_3^{\mathrm{sr}} \mid a(A) - b(A) = 0\}.$$

Similarly, we define subschemes $\mathrm{Mold}_{3,2}^{\mathrm{C}_2 \times \mathrm{D}_1}$ and $\mathrm{Mold}_{3,2}^{S_1}$ of $\mathrm{Mold}_{3,2}$.

How $(\mathrm{C}_2 \times \mathrm{D}_1)$ and S_1 are contained in $\mathrm{Mold}_{3,2}$?

Definition 9

We define an open subscheme $\mathrm{M}_3^{\mathrm{C}_2\times\mathrm{D}_1}$ of $\mathrm{M}_3^{\mathrm{sr}}$ by

$$\mathrm{M}_3^{\mathrm{C}_2\times\mathrm{D}_1} := \{A \in \mathrm{M}_3^{\mathrm{sr}} \mid a(A) - b(A) \neq 0\}.$$

We also define a closed subscheme $\mathrm{M}_3^{S_1}$ of $\mathrm{M}_3^{\mathrm{sr}}$ by

$$\mathrm{M}_3^{S_1} := \{A \in \mathrm{M}_3^{\mathrm{sr}} \mid a(A) - b(A) = 0\}.$$

Similarly, we define subschemes $\mathrm{Mold}_{3,2}^{\mathrm{C}_2\times\mathrm{D}_1}$ and $\mathrm{Mold}_{3,2}^{S_1}$ of $\mathrm{Mold}_{3,2}$. Geometric points of $\mathrm{Mold}_{3,2}^{\mathrm{C}_2\times\mathrm{D}_1}$ and $\mathrm{Mold}_{3,2}^{S_1}$ correspond to subalgebras which are equivalent to $\mathrm{C}_2 \times \mathrm{D}_1$ and S_1, respectively.

How $(\mathrm{C}_2 \times \mathrm{D}_1)$ and S_1 are contained in $\mathrm{Mold}_{3,2}$?

Definition 9

We define an open subscheme $\mathrm{M}_3^{\mathrm{C}_2\times\mathrm{D}_1}$ of $\mathrm{M}_3^{\mathrm{sr}}$ by

$$\mathrm{M}_3^{\mathrm{C}_2\times\mathrm{D}_1} := \{A \in \mathrm{M}_3^{\mathrm{sr}} \mid a(A) - b(A) \neq 0\}.$$

We also define a closed subscheme $\mathrm{M}_3^{S_1}$ of $\mathrm{M}_3^{\mathrm{sr}}$ by

$$\mathrm{M}_3^{S_1} := \{A \in \mathrm{M}_3^{\mathrm{sr}} \mid a(A) - b(A) = 0\}.$$

Similarly, we define subschemes $\mathrm{Mold}_{3,2}^{\mathrm{C}_2\times\mathrm{D}_1}$ and $\mathrm{Mold}_{3,2}^{S_1}$ of $\mathrm{Mold}_{3,2}$. Geometric points of $\mathrm{Mold}_{3,2}^{\mathrm{C}_2\times\mathrm{D}_1}$ and $\mathrm{Mold}_{3,2}^{S_1}$ correspond to subalgebras which are equivalent to $\mathrm{C}_2 \times \mathrm{D}_1$ and S_1, respectively.

Set $\mathrm{Flag} := \{(L, W) \in \mathbb{P}_*(V) \times \mathbb{P}^*(V) \mid L \subset W\} \subset \mathbb{P}_*(V) \times \mathbb{P}^*(V)$.

$\mathrm{Mold}_{3,2}^{\mathrm{C}_2\times\mathrm{D}_1}$ and $\mathrm{Mold}_{3,2}^{S_1}$

$\mathrm{Mold}_{3,2}^{\mathrm{C}_2\times\mathrm{D}_1}$ and $\mathrm{Mold}_{3,2}^{S_1}$

Theorem 10

The isomorphism $\xi : \mathbb{P}_*(V) \times \mathbb{P}^*(V) \to \mathrm{Mold}_{3,2}$ *induces* $\mathrm{Mold}_{3,2}^{\mathrm{C}_2\times\mathrm{D}_1} \cong \mathbb{P}_*(V) \times \mathbb{P}^*(V) \setminus \mathrm{Flag}$ *and* $\mathrm{Mold}_{3,2}^{S_1} \cong \mathrm{Flag}$.

$\mathrm{Mold}_{3,2}^{\mathrm{C}_2\times\mathrm{D}_1}$ and $\mathrm{Mold}_{3,2}^{S_1}$

Theorem 10

The isomorphism $\xi : \mathbb{P}_*(V) \times \mathbb{P}^*(V) \to \mathrm{Mold}_{3,2}$ *induces* $\mathrm{Mold}_{3,2}^{\mathrm{C}_2\times\mathrm{D}_1} \cong \mathbb{P}_*(V) \times \mathbb{P}^*(V) \setminus \mathrm{Flag}$ *and* $\mathrm{Mold}_{3,2}^{S_1} \cong \mathrm{Flag}$. *In particular,* $\mathrm{Mold}_{3,2}^{\mathrm{C}_2\times\mathrm{D}_1}$ *is a smooth integral scheme of relative dimension* 4 *over* $\mathbb{Z}$,

$\mathrm{Mold}_{3,2}^{\mathrm{C}_2 \times \mathrm{D}_1}$ and $\mathrm{Mold}_{3,2}^{S_1}$

Theorem 10

The isomorphism $\xi : \mathbb{P}_*(V) \times \mathbb{P}^*(V) \to \mathrm{Mold}_{3,2}$ *induces* $\mathrm{Mold}_{3,2}^{\mathrm{C}_2 \times \mathrm{D}_1} \cong \mathbb{P}_*(V) \times \mathbb{P}^*(V) \setminus \mathrm{Flag}$ *and* $\mathrm{Mold}_{3,2}^{S_1} \cong \mathrm{Flag}$. *In particular,* $\mathrm{Mold}_{3,2}^{\mathrm{C}_2 \times \mathrm{D}_1}$ *is a smooth integral scheme of relative dimension* 4 *over* $\mathbb{Z}$*, and* $\mathrm{Mold}_{3,2}^{S_1}$ *is a smooth integral scheme of relative dimension* 3 *over* $\mathbb{Z}$*.*

Three-dimensional subalgebras

There are seven types of 3-dimensional k-subalgebras of $\mathrm{M}_3(k)$ over an algebraically closed field k:

Three-dimensional subalgebras

There are seven types of 3-dimensional k-subalgebras of $\mathrm{M}_3(k)$ over an algebraically closed field k:

1. $$\mathrm{D}_3(k) := \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \in \mathrm{M}_3(k) \right\}$$
2. $$(\mathrm{N}_2 \times \mathrm{D}_1)(k) := \left\{ \begin{pmatrix} a & c & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \middle| \, a, b, c \in k \right\}$$
3. $$\mathrm{J}_3(k) := \left\{ \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} \middle| \, a, b, c \in k \right\}$$

1. $S_2(k) := \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & c \\ 0 & 0 & b \end{pmatrix} \middle| \, a, b, c \in k \right\}$
2. $S_3(k) := \left\{ \begin{pmatrix} a & 0 & c \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix} \middle| \, a, b, c \in k \right\}$
3. $S_4(k) := \left\{ \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \middle| \, a, b, c \in k \right\}$
4. $S_5(k) := \left\{ \begin{pmatrix} a & 0 & b \\ 0 & a & c \\ 0 & 0 & a \end{pmatrix} \middle| \, a, b, c \in k \right\}$

The main theorem on $\mathrm{Mold}_{3,3}$

The main theorem on $\mathrm{Mold}_{3,3}$

Theorem 11

There is an irreducible decomposition

$$\mathrm{Mold}_{3,3} = \overline{\mathrm{Mold}_{3,3}^{\mathrm{reg}}} \cup \overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_2}} \cup \overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_3}},$$

where the relative dimensions of $\overline{\mathrm{Mold}_{3,3}^{\mathrm{reg}}}$, $\overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_2}}$, *and* $\overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_3}}$ *over* $\mathbb{Z}$ *are* 6, 4, *and* 4, *respectively.*

The main theorem on $\mathrm{Mold}_{3,3}$

Theorem 11

There is an irreducible decomposition

$$\mathrm{Mold}_{3,3} = \overline{\mathrm{Mold}_{3,3}^{\mathrm{reg}}} \cup \overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_2}} \cup \overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_3}},$$

where the relative dimensions of $\overline{\mathrm{Mold}_{3,3}^{\mathrm{reg}}}$, $\overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_2}}$, *and* $\overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_3}}$ *over* $\mathbb{Z}$ *are* 6, 4, *and* 4, *respectively.*
Moreover, both $\mathrm{Mold}_{3,3}^{\mathrm{S}_5} := \overline{\mathrm{Mold}_{3,3}^{\mathrm{reg}}} \cap \overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_2}}$ *and* $\mathrm{Mold}_{3,3}^{\mathrm{S}_4} := \overline{\mathrm{Mold}_{3,3}^{\mathrm{reg}}} \cap \overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_3}}$ *have relative dimension* 2 *over* $\mathbb{Z}$, *and* $\overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_2}} \cap \overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_3}} = \emptyset$.

The regular part $\mathrm{M}_n^{\mathrm{reg}}$ of M_n

Definition 12

Let M_n be the scheme of $n \times n$-matrices over $\mathbb{Z}$. The scheme M_n is isomorphic to the affine space $\mathbb{A}_{\mathbb{Z}}^{n^2}$.

The regular part $\mathrm{M}_n^{\mathrm{reg}}$ of M_n

Definition 12

Let M_n be the scheme of $n \times n$-matrices over $\mathbb{Z}$. The scheme M_n is isomorphic to the affine space $\mathbb{A}_{\mathbb{Z}}^{n^2}$. Let A be the universal matrix on M_n.

The regular part $\mathrm{M}_n^{\mathrm{reg}}$ of M_n

Definition 12

Let M_n be the scheme of $n \times n$-matrices over $\mathbb{Z}$. The scheme M_n is isomorphic to the affine space $\mathbb{A}_{\mathbb{Z}}^{n^2}$. Let A be the universal matrix on M_n. The open subscheme $\mathrm{M}_n^{\mathrm{reg}}$ of M_n is defined by

$$\mathrm{M}_n^{\mathrm{reg}} := \{x \in \mathrm{M}_n \mid I_n, A, A^2, \ldots, A^{n-1} : \text{linearly independent in } \mathrm{M}_n(k(x))\},$$

where $k(x)$ is the residue field of x. We call $\mathrm{M}_n^{\mathrm{reg}}$ the *regular part* of M_n.

The regular part $\mathrm{M}_n^{\mathrm{reg}}$ of M_n

Definition 12

Let M_n be the scheme of $n \times n$-matrices over $\mathbb{Z}$. The scheme M_n is isomorphic to the affine space $\mathbb{A}_{\mathbb{Z}}^{n^2}$. Let A be the universal matrix on M_n. The open subscheme $\mathrm{M}_n^{\mathrm{reg}}$ of M_n is defined by

$$\mathrm{M}_n^{\mathrm{reg}} := \{x \in \mathrm{M}_n \mid I_n, A, A^2, \ldots, A^{n-1} : \text{linearly independent in } \mathrm{M}_n(k(x))\},$$

where $k(x)$ is the residue field of x. We call $\mathrm{M}_n^{\mathrm{reg}}$ the *regular part* of M_n. For a commutative ring R, we call a matrix $A \in \mathrm{M}_n^{\mathrm{reg}}(R)$ *regular* or *non-derogatory*.

Proposition 1.5

Let R be a local ring. Let $A \in \mathrm{M}_n^{\mathrm{reg}}(R)$.

Proposition 1.5

Let R be a local ring. Let $A \in \mathrm{M}_n^{\mathrm{reg}}(R)$. There exists $P \in \mathrm{GL}_n(R)$ such that

$$P^{-1}AP = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -c_n \\ 1 & 0 & 0 & \ddots & 0 & -c_{n-1} \\ 0 & 1 & 0 & \ddots & 0 & -c_{n-2} \\ 0 & 0 & 1 & \ddots & 0 & -c_{n-3} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -c_1 \end{pmatrix}.$$

Note that $x^n + c_1 x^{n-1} + \cdots + c_{n-1}x + c_n$ is the characteristic polynomial of A.

$\mathrm{M}_n^{\mathrm{reg}}/\mathrm{PGL}_n \cong \mathbb{A}_{\mathbb{Z}}^n$

The group scheme PGL_n acts on $\mathrm{M}_n^{\mathrm{reg}}$ by $A \mapsto P^{-1}AP$.

$\mathrm{M}_n^{\mathrm{reg}}/\mathrm{PGL}_n \cong \mathbb{A}_{\mathbb{Z}}^n$

The group scheme PGL_n acts on $\mathrm{M}_n^{\mathrm{reg}}$ by $A \mapsto P^{-1}AP$.
We define $\pi : \mathrm{M}_n^{\mathrm{reg}} \to \mathbb{A}_{\mathbb{Z}}^n$ by $A \mapsto (c_1, c_2, \ldots, c_n)$, where
$x^n + c_1x^{n-1} + \cdots + c_{n-1}x + c_n$ is the characteristic polynomial of A.

$\mathrm{M}_n^{\mathrm{reg}}/\mathrm{PGL}_n \cong \mathbb{A}_{\mathbb{Z}}^n$

The group scheme PGL_n acts on $\mathrm{M}_n^{\mathrm{reg}}$ by $A \mapsto P^{-1}AP$.
We define $\pi : \mathrm{M}_n^{\mathrm{reg}} \to \mathbb{A}_{\mathbb{Z}}^n$ by $A \mapsto (c_1, c_2, \ldots, c_n)$, where $x^n + c_1 x^{n-1} + \cdots + c_{n-1}x + c_n$ is the characteristic polynomial of A.

Theorem 13

The morphism $\pi : \mathrm{M}_n^{\mathrm{reg}} \to \mathbb{A}_{\mathbb{Z}}^n$ is a universal geometric quotient by PGL_n.

The smooth morphism $\psi : \mathrm{M}_3^{\mathrm{reg}} \to \mathrm{Mold}_{3,3}$

Let $\langle A \rangle$ be the subalgebra generated by A for $A \in \mathrm{M}_3^{\mathrm{reg}}$. We define $\psi : \mathrm{M}_3^{\mathrm{reg}} \to \mathrm{Mold}_{3,3}$ by $A \mapsto \langle A \rangle$.

The smooth morphism $\psi : \mathrm{M}_3^{\mathrm{reg}} \to \mathrm{Mold}_{3,3}$

Let $\langle A \rangle$ be the subalgebra generated by A for $A \in \mathrm{M}_3^{\mathrm{reg}}$. We define $\psi : \mathrm{M}_3^{\mathrm{reg}} \to \mathrm{Mold}_{3,3}$ by $A \mapsto \langle A \rangle$.

Proposition 1.6

The morphism $\psi : \mathrm{M}_3^{\mathrm{reg}} \to \mathrm{Mold}_{3,3}$ is smooth and surjective.

The smooth morphism $\psi : \mathrm{M}_3^{\mathrm{reg}} \to \mathrm{Mold}_{3,3}$

Let $\langle A \rangle$ be the subalgebra generated by A for $A \in \mathrm{M}_3^{\mathrm{reg}}$. We define $\psi : \mathrm{M}_3^{\mathrm{reg}} \to \mathrm{Mold}_{3,3}$ by $A \mapsto \langle A \rangle$.

Proposition 1.6

The morphism $\psi : \mathrm{M}_3^{\mathrm{reg}} \to \mathrm{Mold}_{3,3}$ is smooth and surjective.

Definition 14

We define an open subscheme $\mathrm{Mold}_{3,3}^{\mathrm{reg}}$ of $\mathrm{Mold}_{3,3}$ by $\mathrm{Mold}_{3,3}^{\mathrm{reg}} := \psi(\mathrm{M}_3^{\mathrm{reg}})$.

$\mathrm{Mold}_{3,3}^{\mathrm{D}_3}$, $\mathrm{Mold}_{3,3}^{\mathrm{N}_2 \times \mathrm{D}_2}$, $\mathrm{Mold}_{3,3}^{\mathrm{J}_3}$, and so on

$\mathrm{Mold}_{3,3}^{\mathrm{D}_3}$, $\mathrm{Mold}_{3,3}^{\mathrm{N}_2\times\mathrm{D}_2}$, $\mathrm{Mold}_{3,3}^{\mathrm{J}_3}$, and so on

Theorem 15

The smooth integral scheme $\mathrm{Mold}_{3,3}^{\mathrm{reg}}$ *of relative dimension* 6 *over* $\mathbb{Z}$ *has a stratification of subschemes*

$$\mathrm{Mold}_{3,3}^{\mathrm{reg}} = \mathrm{Mold}_{3,3}^{\mathrm{D}_3} \coprod \mathrm{Mold}_{3,3}^{\mathrm{N}_2\times\mathrm{D}_1} \coprod \mathrm{Mold}_{3,3}^{\mathrm{N}_2\times\mathrm{D}_1/\mathbb{F}_2} \coprod \mathrm{Mold}_{3,3}^{\mathrm{J}_3} \coprod \mathrm{Mold}_{3,3}^{\mathrm{J}_3/\mathbb{F}_3}$$

$\mathrm{Mold}_{3,3}^{\mathrm{D}_3}$, $\mathrm{Mold}_{3,3}^{\mathrm{N}_2\times\mathrm{D}_2}$, $\mathrm{Mold}_{3,3}^{\mathrm{J}_3}$, and so on

Theorem 15

The smooth integral scheme $\mathrm{Mold}_{3,3}^{\mathrm{reg}}$ *of relative dimension* 6 *over* $\mathbb{Z}$ *has a stratification of subschemes*

$$\mathrm{Mold}_{3,3}^{\mathrm{reg}} = \mathrm{Mold}_{3,3}^{\mathrm{D}_3} \coprod \mathrm{Mold}_{3,3}^{\mathrm{N}_2\times\mathrm{D}_1} \coprod \mathrm{Mold}_{3,3}^{\mathrm{N}_2\times\mathrm{D}_1/\mathbb{F}_2} \coprod \mathrm{Mold}_{3,3}^{\mathrm{J}_3} \coprod \mathrm{Mold}_{3,3}^{\mathrm{J}_3/\mathbb{F}_3}$$

such that

1. $\mathrm{Mold}_{3,3}^{\mathrm{D}_3}$ *is a smooth integral scheme of rel. dim.* 6 *over* $\mathbb{Z}$.
2. $\mathrm{Mold}_{3,3}^{\mathrm{N}_2\times\mathrm{D}_1}$ *is a smooth integral scheme of rel. dim.* 5 *over* $\mathbb{Z}[1/2]$.
3. $\mathrm{Mold}_{3,3}^{\mathrm{N}_2\times\mathrm{D}_1/\mathbb{F}_2}$ *is a smooth variety of dimension* 5 *over* $\mathbb{F}_2$.
4. $\mathrm{Mold}_{3,3}^{\mathrm{J}_3}$ *is a smooth integral scheme of rel. dim.* 4 *over* $\mathbb{Z}[1/3]$.
5. $\mathrm{Mold}_{3,3}^{\mathrm{J}_3/\mathbb{F}_3}$ *is a smooth variety of dimension* 4 *over* $\mathbb{F}_3$.

The morphisms $\varphi_{\mathrm{S}_2} : \mathbb{P}^*(V) \times \mathbb{P}^*(V) \to \mathrm{Mold}_{3,3}$ and $\varphi_{\mathrm{S}_3} : \mathbb{P}_*(V) \times \mathbb{P}_*(V) \to \mathrm{Mold}_{3,3}$

The morphisms $\varphi_{\mathrm{S}_2} : \mathbb{P}^*(V) \times \mathbb{P}^*(V) \to \mathrm{Mold}_{3,3}$ and $\varphi_{\mathrm{S}_3} : \mathbb{P}_*(V) \times \mathbb{P}_*(V) \to \mathrm{Mold}_{3,3}$

Let $V = \mathcal{O}_{\mathbb{Z}}^{\oplus 3}$ be a free sheaf of rank 3 on $\operatorname{Spec} \mathbb{Z}$. Let us denote by $\mathbb{P}_*(V)$ and $\mathbb{P}^*(V)$ the projective spaces consisting of rank 1 and rank 2 subbundles of V, respectively.

The morphisms $\varphi_{\mathrm{S}_2} : \mathbb{P}^*(V) \times \mathbb{P}^*(V) \to \mathrm{Mold}_{3,3}$ and $\varphi_{\mathrm{S}_3} : \mathbb{P}_*(V) \times \mathbb{P}_*(V) \to \mathrm{Mold}_{3,3}$

Let $V = \mathcal{O}_{\mathbb{Z}}^{\oplus 3}$ be a free sheaf of rank 3 on $\mathrm{Spec}\,\mathbb{Z}$. Let us denote by $\mathbb{P}_*(V)$ and $\mathbb{P}^*(V)$ the projective spaces consisting of rank 1 and rank 2 subbundles of V, respectively.

Let us define $\varphi_{\mathrm{S}_2} : \mathbb{P}^*(V) \times \mathbb{P}^*(V) \to \mathrm{Mold}_{3,3}$ by $(W_1, W_2) \mapsto \langle \mathrm{Hom}(V/W_1, W_2) \rangle$, where $\langle \mathrm{Hom}(V/W_1, W_2) \rangle$ is the subalgebra of $\mathrm{Hom}(V, V)$ generated by $\mathrm{Hom}(V/W_1, W_2)$.

The morphisms $\varphi_{\mathrm{S}_2} : \mathbb{P}^*(V) \times \mathbb{P}^*(V) \to \mathrm{Mold}_{3,3}$ and $\varphi_{\mathrm{S}_3} : \mathbb{P}_*(V) \times \mathbb{P}_*(V) \to \mathrm{Mold}_{3,3}$

Let $V = \mathcal{O}_{\mathbb{Z}}^{\oplus 3}$ be a free sheaf of rank 3 on $\operatorname{Spec} \mathbb{Z}$. Let us denote by $\mathbb{P}_*(V)$ and $\mathbb{P}^*(V)$ the projective spaces consisting of rank 1 and rank 2 subbundles of V, respectively.

Let us define $\varphi_{\mathrm{S}_2} : \mathbb{P}^*(V) \times \mathbb{P}^*(V) \to \mathrm{Mold}_{3,3}$ by $(W_1, W_2) \mapsto \langle \mathrm{Hom}(V/W_1, W_2) \rangle$, where $\langle \mathrm{Hom}(V/W_1, W_2) \rangle$ is the subalgebra of $\mathrm{Hom}(V, V)$ generated by $\mathrm{Hom}(V/W_1, W_2)$.

Let us define $\varphi_{\mathrm{S}_3} : \mathbb{P}_*(V) \times \mathbb{P}_*(V) \to \mathrm{Mold}_{3,3}$ by $(L_1, L_2) \mapsto \langle \mathrm{Hom}(V/L_1, L_2) \rangle$, where $\langle \mathrm{Hom}(V/L_1, L_2) \rangle$ is the subalgebra of $\mathrm{Hom}(V, V)$ generated by $\mathrm{Hom}(V/L_1, L_2)$.

$\mathrm{Mold}_{3,3}^{\mathrm{S}_2}$ and $\mathrm{Mold}_{3,3}^{\mathrm{S}_3}$

$\mathrm{Mold}_{3,3}^{\mathrm{S}_2}$ and $\mathrm{Mold}_{3,3}^{\mathrm{S}_3}$

Definition 16

We define open subschemes $\mathrm{Mold}_{3,3}^{\mathrm{S}_2}$ and $\mathrm{Mold}_{3,3}^{\mathrm{S}_3}$ of $\mathrm{Mold}_{3,3}$ as
$\mathrm{Mold}_{3,3}^{\mathrm{S}_2} := \varphi_{\mathrm{S}_2}(\mathbb{P}^*(V) \times \mathbb{P}^*(V) \setminus \Delta)$ and
$\mathrm{Mold}_{3,3}^{\mathrm{S}_3} := \varphi_{\mathrm{S}_3}(\mathbb{P}_*(V) \times \mathbb{P}_*(V) \setminus \Delta)$, respectively. Here we denote by Δ the diagonals of $\mathbb{P}^*(V) \times \mathbb{P}^*(V)$ or $\mathbb{P}_*(V) \times \mathbb{P}_*(V)$.

Definition 16

We define open subschemes $\mathrm{Mold}_{3,3}^{\mathrm{S}_2}$ and $\mathrm{Mold}_{3,3}^{\mathrm{S}_3}$ of $\mathrm{Mold}_{3,3}$ as $\mathrm{Mold}_{3,3}^{\mathrm{S}_2} := \varphi_{\mathrm{S}_2}(\mathbb{P}^*(V) \times \mathbb{P}^*(V) \setminus \Delta)$ and $\mathrm{Mold}_{3,3}^{\mathrm{S}_3} := \varphi_{\mathrm{S}_3}(\mathbb{P}_*(V) \times \mathbb{P}_*(V) \setminus \Delta)$, respectively. Here we denote by Δ the diagonals of $\mathbb{P}^*(V) \times \mathbb{P}^*(V)$ or $\mathbb{P}_*(V) \times \mathbb{P}_*(V)$.

Note that

$$\overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_2}} = \varphi_{\mathrm{S}_2}(\mathbb{P}^*(V) \times \mathbb{P}^*(V))$$

and

$$\overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_3}} = \varphi_{\mathrm{S}_3}(\mathbb{P}_*(V) \times \mathbb{P}_*(V)).$$

Again : the main theorem on $\mathrm{Mold}_{3,3}$

Again : the main theorem on $\mathrm{Mold}_{3,3}$

Theorem 17 (Theorem 11)

There is an irreducible decomposition

$$\mathrm{Mold}_{3,3} = \overline{\mathrm{Mold}_{3,3}^{\mathrm{reg}}} \cup \overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_2}} \cup \overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_3}},$$

where the relative dimensions of $\overline{\mathrm{Mold}_{3,3}^{\mathrm{reg}}}$, $\overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_2}}$, *and* $\overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_3}}$ *over* $\mathbb{Z}$ *are* 6, 4, *and* 4, *respectively.*

Again : the main theorem on $\mathrm{Mold}_{3,3}$

Theorem 17 (Theorem 11)

There is an irreducible decomposition

$$\mathrm{Mold}_{3,3} = \overline{\mathrm{Mold}_{3,3}^{\mathrm{reg}}} \cup \overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_2}} \cup \overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_3}},$$

where the relative dimensions of $\overline{\mathrm{Mold}_{3,3}^{\mathrm{reg}}}$, $\overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_2}}$, *and* $\overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_3}}$ *over* $\mathbb{Z}$ *are* 6, 4, *and* 4, *respectively.*
Moreover, both $\mathrm{Mold}_{3,3}^{\mathrm{S}_5} := \overline{\mathrm{Mold}_{3,3}^{\mathrm{reg}}} \cap \overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_2}}$ *and* $\overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_4}} := \overline{\mathrm{Mold}_{3,3}^{\mathrm{reg}}} \cap \overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_3}}$ *have relative dimension* 2 *over* $\mathbb{Z}$, *and* $\overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_2}} \cap \overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_3}} = \emptyset$.

The description of $\mathrm{Mold}_{3,3}$

$$\mathrm{D}_3 = \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \right\}$$

$$|$$

$$\mathrm{N}_2 \times \mathrm{D}_1 = \left\{ \begin{pmatrix} a & c & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \right\}$$

$$|$$

$$\mathrm{S}_2 = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & c \\ 0 & 0 & b \end{pmatrix} \right\} \qquad \mathrm{J}_3 = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} \right\} \qquad \mathrm{S}_3 = \left\{ \begin{pmatrix} a & 0 & c \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix} \right\}$$

$$\backslash \qquad / \qquad \backslash \qquad /$$

$$\mathrm{S}_5 = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & a & c \\ 0 & 0 & a \end{pmatrix} \right\} \qquad \mathrm{S}_4 = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \right\}$$

Appendix (1)

Proposition 1.7

Any subalgebras of $\mathrm{M}_2(k)$ *can be classified into one of the following cases:*

1. $\mathrm{M}_2(k)$
2. $\mathrm{B}_2(k) := \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$
3. $\mathrm{D}_2(k) := \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}$
4. $\mathrm{N}_2(k) := \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \middle| \, a, b \in k \right\}$
5. $\mathrm{C}_2(k) := \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \middle| \, a \in k \right\}$

Appendix (2)

Example 18

In the case $n = 2$, we have

$$\begin{aligned} \mathrm{Mold}_{2,1} &= \mathrm{Spec}\mathbb{Z}, \\ \mathrm{Mold}_{2,2} &= \mathbb{P}^2_{\mathbb{Z}}, \\ \mathrm{Mold}_{2,3} &= \mathbb{P}^1_{\mathbb{Z}}, \\ \mathrm{Mold}_{2,4} &= \mathrm{Spec}\mathbb{Z}. \end{aligned}$$