Central elements of the Jennings basis and certain Morita invariants

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Notation

- F: an algebraically closed field of characteristic $p > 0$
- G: a finite group
- A, B: finite-dimensional (unital associative) algebras over F

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Morita invariants

Theorem 1.1 (S.)

Let A and B be Morita equivalent. Then there is an algebra isomorphism

$$Z(A) \to Z(B)$$

mapping $ZS^n(A)$ onto $ZS^n(B)$ for every $n \in \mathbb{N}$. In particular, $ZS^n(A)$ are Morita invariants.

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Namely, the dimension of $ZS^n(A)$ could be described representation-theoretically as well.

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Jennings theory

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Theorem 2.1 (Jennings)

There are elements $\{\, g_{ij} \in G \mid 1 \leq i < t,\ 1 \leq j \leq r_i \,\}$ *such that*

$$\mathrm{Soc}^n(FG) = \bigoplus F \prod_{\substack{1 \leq i < t \\ 1 \leq j \leq r_i}}{}' (g_{ij} - 1)^{m_{ij}}$$

for every integer $n \geq 0$ *where the direct sum is taken for all integers* $0 \leq m_{ij} < p$ *satisfying* $\sum_{\substack{1 \leq i < t \\ 1 \leq j \leq r_i}} i(p - 1 - m_{ij}) < n$.

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Definition 2.3 (Jennings basis)

The basis $\{ \prod'_{\substack{1 \leq i < t \\ 1 \leq j \leq r_i}} (g_{ij} - 1)^{m_{ij}} \}$ of FG is said to be the *Jennings basis*.

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Example

Let G be an extra-special p-group of order p^3 and exponent $p > 2$ defined by

$$G := p_+^{1+2} = \langle a, b, c \mid a^p = b^p = c^p = [a, c] = [b, c] = 1,\ [b, a] = c \rangle$$

and set $x := a - 1$, $y := b - 1$, and $z := c - 1$.

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$$\{ x^i y^j z^k \mid 0 \leq i, j, k < p \}$$

is a Jennings basis of FG and we can show

$$ZS^n(FG) = \bigoplus_{\substack{0 \leq i,j < p \\ 2(p-1)-(i+j) < n}} F x^i y^j z^{p-1} \oplus \bigoplus_{\substack{0 \leq k < p-1 \\ 4(p-1)-2k < n}} F z^k.$$

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Theorem 4.1 (S.)

Suppose $s \in \mathbb{N}$ satisfies $D_s \geq [G,G]$. Then an element of the Jennings basis of the form

$$\prod'_{\substack{1 \leq i < s \\ 1 \leq j \leq r_i}} (g_{ij} - 1)^{m_{ij}} \prod'_{\substack{s \leq i < t \\ 1 \leq j \leq r_i}} (g_{ij} - 1)^{p-1}$$

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is central for every integers $0 \leq m_{ij} < p$. In particular, for every $n \in \mathbb{N}$, we have

$$ZS^n(FG) \supseteq \bigoplus F \prod_{\substack{1 \leq i < s \\ 1 \leq j \leq r_i}}' (g_{ij}-1)^{m_{ij}} \prod_{\substack{s \leq i < t \\ 1 \leq j \leq r_i}}' (g_{ij}-1)^{p-1}$$

where the direct sum is taken for all integers $0 \leq m_{ij} < p$ satisfying

$$\sum_{\substack{1 \leq i < s \\ 1 \leq j \leq r_i}} i(p-1-m_{ij}) < n.$$