Around Azumaya rings - An overview of ring theory in the last decades

Robert Wisbauer

University of Düsseldorf, Germany

Yamanashi, October 2017

Overview

- 19 th Century, Hilbert - 1900

Overview

- 19 th Century, Hilbert - 1900
- Wedderburn structure theorems - 1908

Overview

- 19 th Century, Hilbert - 1900
- Wedderburn structure theorems - 1908
- Category theory - 1945

Overview

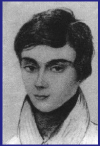
- 19 th Century, Hilbert - 1900
- Wedderburn structure theorems - 1908
- Category theory - 1945
- Equivalence of categories (Morita, Azumaya, Hopf)

Overview

- 19 th Century, Hilbert - 1900
- Wedderburn structure theorems - 1908
- Category theory - 1945
- Equivalence of categories (Morita, Azumaya, Hopf)
- Separability revisited

19 th century

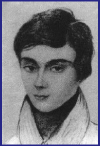
19 th century



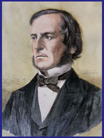
Evariste Galois, 1811 - 1832
Paris, France

Field extensions, group theory

19 th century

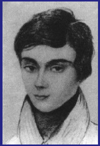


Evariste Galois, 1811 - 1832
Paris, France
Field extensions, group theory

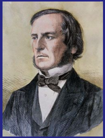


George Boole, 1815 - 1864
England
The Mathematical Analysis of Logic 1847

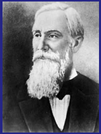
19 th century



Evariste Galois, 1811 - 1832
Paris, France
Field extensions, group theory



George Boole, 1815 - 1864
England
The Mathematical Analysis of Logic 1847



Pafnuty Chebyshev, 1821 - 1894
Skt. Petersburg, Russia
Theorie der Kongruenzen, Berlin 1889

19 th century

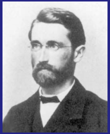
Leopold Kronecker, 1823 - 1891

Berlin, Germany

Grundzüge einer arithmetischen Theorie der algebraischen Grössen, 1882

19 th century

Leopold Kronecker, 1823 - 1891
Berlin, Germany
Grundzüge einer arithmetischen Theorie der algebraischen Grössen, 1882

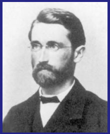


Richard Dedekind, 1831 - 1916
Braunschweig, Germany
On the theory of algebraic integers, 1877

19 th century

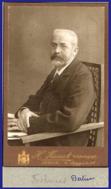
Leopold Kronecker, 1823 - 1891
Berlin, Germany

Grundzüge einer arithmetischen Theorie der algebraischen Grössen, 1882



Richard Dedekind, 1831 - 1916
Braunschweig, Germany

On the theory of algebraic integers, 1877



Ferdinand Frobenius, 1849 - 1917
Berlin, Germany

Theorie der hyperkomplexen Größen, 1903

19 th century - Japan

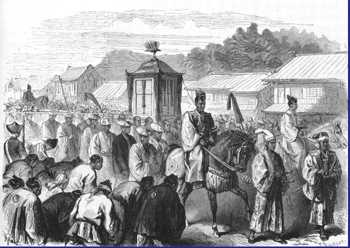
Samurai der Shimazu
Satsuma

Boshin war period, 1868 -1869

19 th century - Japan

Samurai der Shimazu
Satsuma

Boshin war period, 1868 -1869



Meiji restoration
Edo period → Imperial period

Meiji Emperor moves
from Kyoto to Tokio, 1868

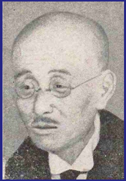
19 th century - Japan

Dairoku Kikuchi, 1855 - 1917, Tokyo

sent to England 1866, Cambridge 1870 - 1877,
Tokyo University, textbook "Elementary Geometry"

19 th century - Japan

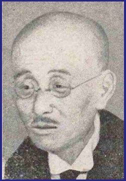
Dairoku Kikuchi, 1855 - 1917, Tokyo
sent to England 1866, Cambridge 1870 - 1877,
Tokyo University, textbook "Elementary Geometry"



Rikitaro Fujisawa, 1861 - 1933, Tokyo
England-Germany 1883 - 1887, lectures by Kronecker
Ph.D. Strasbourg 1886 (Fourier series, E. Christoffel)
talk at the Internat. Math. Congress in Paris, 1900

19 th century - Japan

Dairoku Kikuchi, 1855 - 1917, Tokyo
sent to England 1866, Cambridge 1870 - 1877,
Tokyo University, textbook "Elementary Geometry"



Rikitaro Fujisawa, 1861 - 1933, Tokyo
England-Germany 1883 - 1887, lectures by Kronecker
Ph.D. Strasbourg 1886 (Fourier series, E. Christoffel)
talk at the Internat. Math. Congress in Paris, 1900

Teiji Takagi, 1875 - 1960, Tokyo
Class Field Theory, Ph.D. Tokyo 1903
teacher of ***Goro Azumaya*** *in class field theory*

19 - 20 th century

David Hilbert, 1862 - 1943, Göttingen

Über den Zahlbegriff

Jahresbericht Deutsch. Math. Ver., 1900

19 - 20 th century

David Hilbert, 1862 - 1943, Göttingen

Über den Zahlbegriff

Jahresbericht Deutsch. Math. Ver., 1900

Geometry: axioms for Euclidean plane

Real numbers: natural numbers, integers, ...(genetic method)

David Hilbert, 1862 - 1943, Göttingen

Über den Zahlbegriff

Jahresbericht Deutsch. Math. Ver., 1900

Geometry: axioms for Euclidean plane

Real numbers: natural numbers, integers, ...(genetic method)

My opinion is this: in spite of the high pedagogical and heuristic value of the genetic method, for the final presentation and full logical assurance of the content of our knowledge, the axiomatic method deserves preference.

19 - 20 th century

David Hilbert, 1862 - 1943, Göttingen

Über den Zahlbegriff

Jahresbericht Deutsch. Math. Ver., 1900

Geometry: axioms for Euclidean plane

Real numbers: natural numbers, integers, ...(genetic method)

My opinion is this: in spite of the high pedagogical and heuristic value of the genetic method, for the final presentation and full logical assurance of the content of our knowledge, the axiomatic method deserves preference.

Axioms: $(R, +, \cdot)$, ring, field, order, Archimedean, completeness

19 - 20 th century

Joseph Wedderburn, 1882 - 1948, Princeton

On hypercomplex numbers

Proc. London Math. Soc. 1908

19 - 20 th century

Joseph Wedderburn, 1882 - 1948, Princeton
On hypercomplex numbers
Proc. London Math. Soc. 1908

Finite dimensional algebras A over fields

(i) A has a maximal nilpotent ideal N (radical);

19 - 20 th century

Joseph Wedderburn, 1882 - 1948, Princeton
On hypercomplex numbers
Proc. London Math. Soc. 1908

Finite dimensional algebras A over fields

(i) A has a maximal nilpotent ideal N (radical);

(ii) if $N = 0$, then A is a direct sum of simple matric algebras;

19 - 20 th century

Joseph Wedderburn, 1882 - 1948, Princeton
On hypercomplex numbers
Proc. London Math. Soc. 1908

Finite dimensional algebras A over fields

(i) A has a maximal nilpotent ideal N (radical);

(ii) if $N = 0$, then A is a direct sum of simple matric algebras;

(iii) A simple: matric algebra over division ring;

19 - 20 th century

Joseph Wedderburn, 1882 - 1948, Princeton
On hypercomplex numbers
Proc. London Math. Soc. 1908

Finite dimensional algebras A over fields

(i) A has a maximal nilpotent ideal N (radical);
(ii) if $N = 0$, then A is a direct sum of simple matric algebras;
(iii) A simple: matric algebra over division ring;
(iv) $A = B + N$, B semisimple and N nilpotent (structure?);

Joseph Wedderburn, 1882 - 1948, Princeton

On hypercomplex numbers

Proc. London Math. Soc. 1908

Finite dimensional algebras A over fields

(i) A has a maximal nilpotent ideal N (radical);

(ii) if $N = 0$, then A is a direct sum of simple matric algebras;

(iii) A simple: matric algebra over division ring;

(iv) $A = B + N$, B semisimple and N nilpotent (structure?);

(v) not invertible elements nilpotent: $A = B + N$, B div. ring;

Joseph Wedderburn, 1882 - 1948, Princeton
On hypercomplex numbers
Proc. London Math. Soc. 1908

Finite dimensional algebras A over fields

(i) A has a maximal nilpotent ideal N (radical);
(ii) if $N = 0$, then A is a direct sum of simple matric algebras;
(iii) A simple: matric algebra over division ring;
(iv) $A = B + N$, B semisimple and N nilpotent (structure?);
(v) not invertible elements nilpotent: $A = B + N$, B div. ring;
(vi) A not associative: associative subalgebras nucleus, centre.

19 - 20 th century

Joseph Wedderburn, 1882 - 1948, Princeton
On hypercomplex numbers
Proc. London Math. Soc. 1908

Finite dimensional algebras A over fields

(i) A has a maximal nilpotent ideal N (radical);
(ii) if $N = 0$, then A is a direct sum of simple matric algebras;
(iii) A simple: matric algebra over division ring;
(iv) $A = B + N$, B semisimple and N nilpotent (structure?);
(v) not invertible elements nilpotent: $A = B + N$, B div. ring;
(vi) A not associative: associative subalgebras nucleus, centre.

The classification of algebras cannot be carried out much further than this till a classification of nilpotent algebras has been found ...

20 th century

Representation theory

quiver representations, path algebras, Auslander-Reiten theory, tilting theory, bocses

20 th century

Representation theory

quiver representations, path algebras, Auslander-Reiten theory, tilting theory, bocses

Module theory

Characterisation of rings by properties of their modules:
Homological classification of rings (L.A. Skornjakov, 1967),

20 th century

Representation theory

quiver representations, path algebras, Auslander-Reiten theory, tilting theory, bocses

Module theory

Characterisation of rings by properties of their modules: *Homological classification of rings (L.A. Skornjakov, 1967)*,

Radical theory

"semisimple" and "radical" classes of rings, Jacobson radical, Brown-McCoy radical, Kurosch-Amitsur radical, torsion theory

20 th century

Representation theory

quiver representations, path algebras, Auslander-Reiten theory, tilting theory, bocses

Module theory

Characterisation of rings by properties of their modules: *Homological classification of rings (L.A. Skornjakov, 1967)*,

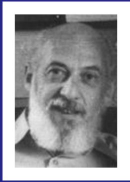
Radical theory

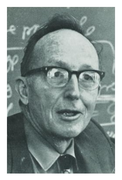
"semisimple" and "radical" classes of rings, Jacobson radical, Brown-McCoy radical, Kurosch-Amitsur radical, torsion theory

Non-associative algebras

alternative algebras, Jordan algebras, Lie algebras, Malcev algebras, Leibniz algebras

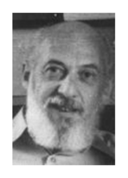
Category theory

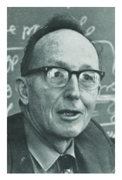




Eilenberg - Mac Lane,
General Theory of natural equivalences
Trans. AMS 1945

Category theory

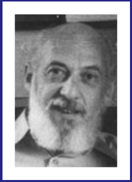


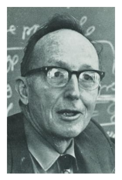


Eilenberg - Mac Lane,
General Theory of natural equivalences
Trans. AMS 1945

Category $\mathbb{A}$: objects and morphisms $\mathrm{Mor}_{\mathbb{A}}(A, A')$,

Category theory



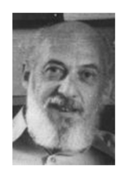
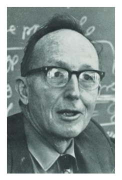


Eilenberg - Mac Lane,
General Theory of natural equivalences
Trans. AMS 1945

Category $\mathbb{A}$: objects and morphisms $\mathrm{Mor}_{\mathbb{A}}(A, A')$,
adjoint pair of functors $F : \mathbb{A} \to \mathbb{B}$, $\mathbb{G} : \mathbb{B} \to \mathbb{A}$, bijection

$$\mathrm{Mor}_{\mathbb{B}}(F(A), B) \simeq \mathrm{Mor}_{\mathbb{A}}(A, G(B)),$$

Category theory

Eilenberg - Mac Lane,
General Theory of natural equivalences
Trans. AMS 1945

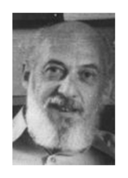
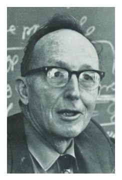
Category $\mathbb{A}$: objects and morphisms $\mathrm{Mor}_{\mathbb{A}}(A, A')$,
adjoint pair of functors $F : \mathbb{A} \to \mathbb{B}$, $\mathbb{G} : \mathbb{B} \to \mathbb{A}$, bijection

$$\mathrm{Mor}_{\mathbb{B}}(F(A), B) \simeq \mathrm{Mor}_{\mathbb{A}}(A, G(B)),$$

unit and counit (nat. transf.) $\eta : 1 \to GF, \quad \varepsilon : FG \to 1$

$$F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F = 1_F, \quad G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G = 1_G$$

Category theory

Eilenberg - Mac Lane,
General Theory of natural equivalences
Trans. AMS 1945

Category $\mathbb{A}$: objects and morphisms $\mathrm{Mor}_{\mathbb{A}}(A, A')$,
adjoint pair of functors $F : \mathbb{A} \to \mathbb{B}$, $\mathbb{G} : \mathbb{B} \to \mathbb{A}$, bijection

$$\mathrm{Mor}_{\mathbb{B}}(F(A), B) \simeq \mathrm{Mor}_{\mathbb{A}}(A, G(B)),$$

unit and counit (nat. transf.) $\eta : 1 \to GF, \quad \varepsilon : FG \to 1$

$$F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F = 1_F, \quad G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G = 1_G$$

Equivalence: η and ε are natural isomorphisms

Application of category theory

Kiiti Morita, 1915 - 1995, Tsukuba, Tokio
Duality for modules and its application ...., 1958

adjoint pair of functors

Application of category theory

Kiiti Morita, 1915 - 1995, Tsukuba, Tokio
Duality for modules and its application ...., 1958

adjoint pair of functors

$$_RM \otimes_S - : {}_S\mathbb{M} \to {}_R\mathbb{M}, \quad \mathrm{Hom}_R(M, -) : {}_R\mathbb{M} \to {}_S\mathbb{M}$$

$$\text{counit} \quad \varepsilon_X : M \otimes_S \mathrm{Hom}_R(M, X) \to X, \; m \otimes f \mapsto f(m),$$

$$\text{unit} \quad \eta_Y : Y \to \mathrm{Hom}_R(M, M \otimes_S Y), \; y \mapsto [m \mapsto m \otimes y]$$

Application of category theory

Kiiti Morita, 1915 - 1995, Tsukuba, Tokio
Duality for modules and its application ...., 1958

adjoint pair of functors

$$ {}_R M \otimes_S - : {}_S\mathbb{M} \to {}_R\mathbb{M}, \quad \mathrm{Hom}_R(M,-) : {}_R\mathbb{M} \to {}_S\mathbb{M} $$

counit $\varepsilon_X : M \otimes_S \mathrm{Hom}_R(M,X) \to X, \; m \otimes f \mapsto f(m),$

unit $\eta_Y : Y \to \mathrm{Hom}_R(M, M \otimes_S Y), \; y \mapsto [m \mapsto m \otimes y]$

Equivalence

${}_R M$ fin. gen. projective, generator, M_S fin. gen. projective,

$$ \mathrm{Hom}_R(M,-) = \mathrm{Hom}_R(M,R) \otimes_R -, \quad R \simeq \mathrm{End}(M_S) $$

Application of category theory

Kiiti Morita, 1915 - 1995, Tsukuba, Tokio
Duality for modules and its application ...., 1958

adjoint pair of functors

$$_RM \otimes_S - : {}_S\mathbb{M} \to {}_R\mathbb{M}, \quad \mathrm{Hom}_R(M, -) : {}_R\mathbb{M} \to {}_S\mathbb{M}$$

$$\text{counit} \quad \varepsilon_X : M \otimes_S \mathrm{Hom}_R(M, X) \to X, \; m \otimes f \mapsto f(m),$$
$$\text{unit} \quad \eta_Y : Y \to \mathrm{Hom}_R(M, M \otimes_S Y), \; y \mapsto [m \mapsto m \otimes y]$$

Equivalence

$_RM$ fin. gen. projective, generator, M_S fin. gen. projective,

$$\mathrm{Hom}_R(M, -) = \mathrm{Hom}_R(M, R) \otimes_R -, \quad R \simeq \mathrm{End}(M_S)$$

$$\text{counit} \quad \varepsilon_X : M \otimes_S M^* \otimes_R X \to X,$$
$$\text{unit} \quad \eta_Y : Y \to M^* \otimes_R M \otimes_S Y$$

Subcategories

$$\mathrm{Gen}(M) = \quad \{N \in {}_R\mathbb{M} \mid M^{(\Lambda)} \to N \to 0\},$$

Subcategories

$$\mathrm{Gen}(M) = \{N \in {}_R\mathbb{M} \mid M^{(\Lambda)} \to N \to 0\},$$
$$\mathrm{Pres}(M) = \{N \in {}_R\mathbb{M} \mid M^{(\Lambda')} \to M^{(\Lambda)} \to N \to 0\},$$

Subcategories

$$\begin{aligned}
\mathrm{Gen}(M) = &\quad \{N \in {}_R\mathbb{M} \mid M^{(\Lambda)} \to N \to 0\},\\
\mathrm{Pres}(M) = &\quad \{N \in {}_R\mathbb{M} \mid M^{(\Lambda')} \to M^{(\Lambda)} \to N \to 0\},\\
\sigma[M] = &\quad \{N \in {}_R\mathbb{M} \mid N \subseteq L,\ L \in \mathrm{Gen}(M)\} = \overline{\mathrm{Gen}(M)}
\end{aligned}$$

Subcategories

$$\begin{aligned} \mathrm{Gen}(M) = &\quad \{N \in {}_R\mathbb{M} \mid M^{(\Lambda)} \to N \to 0\}, \\ \mathrm{Pres}(M) = &\quad \{N \in {}_R\mathbb{M} \mid M^{(\Lambda')} \to M^{(\Lambda)} \to N \to 0\}, \\ \sigma[M] = &\quad \{N \in {}_R\mathbb{M} \mid N \subseteq L,\ L \in \mathrm{Gen}(M)\} = \overline{\mathrm{Gen}(M)} \end{aligned}$$

Q cogenerator in $\mathrm{Gen}(M)$, $U = \mathrm{Hom}_R(M, Q) \in {}_S\mathbb{M}$

$$\mathrm{Cog}(U) = \{L \in {}_S\mathbb{M} \mid L\ U\text{-cog.}\},\ \mathrm{Hom}_R(M, -) : \sigma[M] \to Cog(U)$$

Subcategories

$$\begin{aligned} \mathrm{Gen}(M) = & \; \{N \in {}_R\mathbb{M} \mid M^{(\Lambda)} \to N \to 0\}, \\ \mathrm{Pres}(M) = & \; \{N \in {}_R\mathbb{M} \mid M^{(\Lambda')} \to M^{(\Lambda)} \to N \to 0\}, \\ \sigma[M] = & \; \{N \in {}_R\mathbb{M} \mid N \subseteq L,\ L \in \mathrm{Gen}(M)\} = \overline{\mathrm{Gen}(M)} \end{aligned}$$

Q cogenerator in $\mathrm{Gen}(M)$, $U = \mathrm{Hom}_R(M, Q) \in {}_S\mathbb{M}$

$$\mathrm{Cog}(U) = \{L \in {}_S\mathbb{M} \mid L \ U\text{-cog.}\},\ \mathrm{Hom}_R(M, -) : \sigma[M] \to Cog(U)$$

Adjoint situations

$M \otimes_S - : {}_S\mathbb{M} \to \sigma[M]$, $\mathrm{Hom}_R(M, -) : \sigma[M] \to {}_S\mathbb{M}$,

Subcategories

$$\begin{aligned} \mathrm{Gen}(M) = &\quad \{N \in {}_R\mathbb{M} \mid M^{(\Lambda)} \to N \to 0\}, \\ \mathrm{Pres}(M) = &\quad \{N \in {}_R\mathbb{M} \mid M^{(\Lambda')} \to M^{(\Lambda)} \to N \to 0\}, \\ \sigma[M] = &\quad \{N \in {}_R\mathbb{M} \mid N \subseteq L,\ L \in \mathrm{Gen}(M)\} = \overline{\mathrm{Gen}(M)} \end{aligned}$$

Q cogenerator in $\mathrm{Gen}(M)$, $U = \mathrm{Hom}_R(M, Q) \in {}_S\mathbb{M}$

$$\mathrm{Cog}(U) = \{L \in {}_S\mathbb{M} \mid L \ U\text{-cog.}\},\ \mathrm{Hom}_R(M, -) : \sigma[M] \to Cog(U)$$

Adjoint situations

$$\begin{aligned} &M \otimes_S - : {}_S\mathbb{M} \to \sigma[M], && \mathrm{Hom}_R(M, -) : \sigma[M] \to {}_S\mathbb{M}, \\ &M \otimes_S - : {}_S\mathbb{M} \to \mathrm{Pres}(M), && \mathrm{Hom}_R(M, -) : \mathrm{Pres}(M) \to {}_S\mathbb{M}, \end{aligned}$$

Subcategories

$$\begin{aligned}
\mathrm{Gen}(M) &= \{N \in {}_R\mathbb{M} \mid M^{(\Lambda)} \to N \to 0\}, \\
\mathrm{Pres}(M) &= \{N \in {}_R\mathbb{M} \mid M^{(\Lambda')} \to M^{(\Lambda)} \to N \to 0\}, \\
\sigma[M] &= \{N \in {}_R\mathbb{M} \mid N \subseteq L,\ L \in \mathrm{Gen}(M)\} = \overline{\mathrm{Gen}(M)}
\end{aligned}$$

Q cogenerator in $\mathrm{Gen}(M)$, $U = \mathrm{Hom}_R(M, Q) \in {}_S\mathbb{M}$

$$\mathrm{Cog}(U) = \{L \in {}_S\mathbb{M} \mid L\ U\text{-cog.}\},\ \mathrm{Hom}_R(M, -) : \sigma[M] \to Cog(U)$$

Adjoint situations

$$\begin{aligned}
&M \otimes_S - : {}_S\mathbb{M} \to \sigma[M], && \mathrm{Hom}_R(M, -) : \sigma[M] \to {}_S\mathbb{M}, \\
&M \otimes_S - : {}_S\mathbb{M} \to \mathrm{Pres}(M), && \mathrm{Hom}_R(M, -) : \mathrm{Pres}(M) \to {}_S\mathbb{M}, \\
&M \otimes_S - : \mathrm{Cog}(U) \to \mathrm{Gen}(M), && \mathrm{Hom}_R(M, -) : \mathrm{Gen}(M) \to \mathrm{Cog}(U)
\end{aligned}$$

Subcategories

$$\begin{aligned} \mathrm{Gen}(M) &= \{N \in {}_R\mathbb{M} \mid M^{(\Lambda)} \to N \to 0\}, \\ \mathrm{Pres}(M) &= \{N \in {}_R\mathbb{M} \mid M^{(\Lambda')} \to M^{(\Lambda)} \to N \to 0\}, \\ \sigma[M] &= \{N \in {}_R\mathbb{M} \mid N \subseteq L,\ L \in \mathrm{Gen}(M)\} = \overline{\mathrm{Gen}(M)} \end{aligned}$$

Q cogenerator in $\mathrm{Gen}(M)$, $U = \mathrm{Hom}_R(M, Q) \in {}_S\mathbb{M}$

$$\mathrm{Cog}(U) = \{L \in {}_S\mathbb{M} \mid L\ U\text{-cog.}\},\ \mathrm{Hom}_R(M, -) : \sigma[M] \to Cog(U)$$

Adjoint situations

$$\begin{aligned} &M \otimes_S - : {}_S\mathbb{M} \to \sigma[M], && \mathrm{Hom}_R(M, -) : \sigma[M] \to {}_S\mathbb{M}, \\ &M \otimes_S - : {}_S\mathbb{M} \to \mathrm{Pres}(M), && \mathrm{Hom}_R(M, -) : \mathrm{Pres}(M) \to {}_S\mathbb{M}, \\ &M \otimes_S - : \mathrm{Cog}(U) \to \mathrm{Gen}(M), && \mathrm{Hom}_R(M, -) : \mathrm{Gen}(M) \to \mathrm{Cog}(U) \end{aligned}$$

Equivalences

Fuller's Theorem $\qquad$ M f.gen., self-projective, self-generator

Subcategories

$$\begin{aligned} \mathrm{Gen}(M) = &\quad \{N \in {}_R\mathbb{M} \mid M^{(\Lambda)} \to N \to 0\}, \\ \mathrm{Pres}(M) = &\quad \{N \in {}_R\mathbb{M} \mid M^{(\Lambda')} \to M^{(\Lambda)} \to N \to 0\}, \\ \sigma[M] = &\quad \{N \in {}_R\mathbb{M} \mid N \subseteq L,\ L \in \mathrm{Gen}(M)\} = \overline{\mathrm{Gen}(M)} \end{aligned}$$

Q cogenerator in $\mathrm{Gen}(M)$, $U = \mathrm{Hom}_R(M, Q) \in {}_S\mathbb{M}$

$$\mathrm{Cog}(U) = \{L \in {}_S\mathbb{M} \mid L \ U\text{-cog.}\},\ \mathrm{Hom}_R(M, -) : \sigma[M] \to Cog(U)$$

Adjoint situations

$$\begin{aligned} &M \otimes_S - : {}_S\mathbb{M} \to \sigma[M], && \mathrm{Hom}_R(M, -) : \sigma[M] \to {}_S\mathbb{M}, \\ &M \otimes_S - : {}_S\mathbb{M} \to \mathrm{Pres}(M), && \mathrm{Hom}_R(M, -) : \mathrm{Pres}(M) \to {}_S\mathbb{M}, \\ &M \otimes_S - : \mathrm{Cog}(U) \to \mathrm{Gen}(M), && \mathrm{Hom}_R(M, -) : \mathrm{Gen}(M) \to \mathrm{Cog}(U) \end{aligned}$$

Equivalences

Fuller's Theorem	M f.gen., self-projective, self-generator
Sato's Theorem	M self-small and s-Σ-quasi-projective

Subcategories

$$\begin{aligned} \mathrm{Gen}(M) = & \{N \in {}_R\mathbb{M} \mid M^{(\Lambda)} \to N \to 0\}, \\ \mathrm{Pres}(M) = & \{N \in {}_R\mathbb{M} \mid M^{(\Lambda')} \to M^{(\Lambda)} \to N \to 0\}, \\ \sigma[M] = & \{N \in {}_R\mathbb{M} \mid N \subseteq L,\ L \in \mathrm{Gen}(M)\} = \overline{\mathrm{Gen}(M)} \end{aligned}$$

Q cogenerator in $\mathrm{Gen}(M)$, $U = \mathrm{Hom}_R(M, Q) \in {}_S\mathbb{M}$

$$\mathrm{Cog}(U) = \{L \in {}_S\mathbb{M} \mid L\ U\text{-cog.}\},\ \mathrm{Hom}_R(M, -) : \sigma[M] \to Cog(U)$$

Adjoint situations

$$\begin{aligned} &M \otimes_S - : {}_S\mathbb{M} \to \sigma[M], && \mathrm{Hom}_R(M, -) : \sigma[M] \to {}_S\mathbb{M}, \\ &M \otimes_S - : {}_S\mathbb{M} \to \mathrm{Pres}(M), && \mathrm{Hom}_R(M, -) : \mathrm{Pres}(M) \to {}_S\mathbb{M}, \\ &M \otimes_S - : \mathrm{Cog}(U) \to \mathrm{Gen}(M), && \mathrm{Hom}_R(M, -) : \mathrm{Gen}(M) \to \mathrm{Cog}(U) \end{aligned}$$

Equivalences

Fuller's Theorem	M f.gen., self-projective, self-generator
Sato's Theorem	M self-small and s-Σ-quasi-projective
Menini-Orsatti's Th.	M $*$-module (self-small tilting in $\sigma[M]$)

Category theory

Pierre Gabriel, 1933 - 2015, ETH Zürich

Des categories abeliennes, thesis 1960 :
Grothendieck categories: enough injectives, localisation

Category theory

Pierre Gabriel, 1933 - 2015, ETH Zürich

Des categories abeliennes, thesis 1960 :
Grothendieck categories: enough injectives, localisation

$\sigma[M]$ is Grothendieck category

- M generator in $\sigma[M] \Rightarrow M_S$ flat and $R \to \mathrm{End}(M_S)$ dense;

Category theory

Pierre Gabriel, 1933 - 2015, ETH Zürich

Des categories abeliennes, thesis 1960 :
Grothendieck categories: enough injectives, localisation

$\sigma[M]$ is Grothendieck category

- M generator in $\sigma[M] \Rightarrow M_S$ flat and $R \to \mathrm{End}(M_S)$ dense;
- ${}_RM$ faithful and M_S finitely generated $\Rightarrow \sigma[M] = {}_R\mathbb{M}$;

Category theory

Pierre Gabriel, 1933 - 2015, ETH Zürich

Des categories abeliennes, thesis 1960 :
Grothendieck categories: enough injectives, localisation

$\sigma[M]$ is Grothendieck category

- M generator in $\sigma[M] \Rightarrow M_S$ flat and $R \to \mathrm{End}(M_S)$ dense;
- ${}_RM$ faithful and M_S finitely generated $\Rightarrow \sigma[M] = {}_R\mathbb{M}$;
- M injective in $\sigma[M] \Leftrightarrow M$ is self-injective (Baer Lemma);

Category theory

Pierre Gabriel, 1933 - 2015, ETH Zürich

Des categories abeliennes, thesis 1960 :
Grothendieck categories: enough injectives, localisation

$\sigma[M]$ is Grothendieck category

- M generator in $\sigma[M] \Rightarrow M_S$ flat and $R \to \mathrm{End}(M_S)$ dense;
- ${}_RM$ faithful and M_S finitely generated $\Rightarrow \sigma[M] = {}_R\mathbb{M}$;
- M injective in $\sigma[M] \Leftrightarrow M$ is self-injective (Baer Lemma);
- every injective module in $\sigma[M]$ is M-generated;

Category theory

Pierre Gabriel, 1933 - 2015, ETH Zürich

Des categories abeliennes, thesis 1960 :
Grothendieck categories: enough injectives, localisation

$\sigma[M]$ is Grothendieck category

- M generator in $\sigma[M] \Rightarrow M_S$ flat and $R \to \mathrm{End}(M_S)$ dense;
- ${}_RM$ faithful and M_S finitely generated $\Rightarrow \sigma[M] = {}_R\mathbb{M}$;
- M injective in $\sigma[M] \Leftrightarrow M$ is self-injective (Baer Lemma);
- every injective module in $\sigma[M]$ is M-generated;
- M semisimple $\Leftrightarrow$ every module in $\sigma[M]$ is M-injective;

Category theory

Pierre Gabriel, 1933 - 2015, ETH Zürich

Des categories abeliennes, thesis 1960 :
Grothendieck categories: enough injectives, localisation

$\sigma[M]$ is Grothendieck category

- M generator in $\sigma[M] \Rightarrow M_S$ flat and $R \to \mathrm{End}(M_S)$ dense;
- ${}_RM$ faithful and M_S finitely generated $\Rightarrow \sigma[M] = {}_R\mathbb{M}$;
- M injective in $\sigma[M] \Leftrightarrow M$ is self-injective (Baer Lemma);
- every injective module in $\sigma[M]$ is M-generated;
- M semisimple $\Leftrightarrow$ every module in $\sigma[M]$ is M-injective;
- M finite length: finite repres. type $\Leftrightarrow$ bounded repres. type

Multiplication algebra $M(A)$

$$\lambda_a : A \to A,\ x \mapsto ax; \quad \varrho_a : A \to A,\ x \mapsto xa;$$

$$M(A) := < \{\lambda_a, \varrho_a \,|\, a \in A\} > \subset \mathrm{End}_{\mathbb{Z}}(A)$$

- A is $M(A)$-module, generated by 1_A,

Category theory

Multiplication algebra $M(A)$

$$\lambda_a : A \to A,\ x \mapsto ax; \quad \varrho_a : A \to A,\ x \mapsto xa;$$

$$M(A) := < \{\lambda_a, \varrho_a \mid a \in A\} > \subset \mathrm{End}_{\mathbb{Z}}(A)$$

- A is $M(A)$-module, generated by 1_A,
- $\mathrm{End}_{M(A)}(A) = Z(A)$ (center), $\quad M(A) \subset \mathrm{End}(A_{Z(A)})$,

Multiplication algebra $M(A)$

$$\lambda_a : A \to A,\ x \mapsto ax; \quad \varrho_a : A \to A,\ x \mapsto xa;$$

$$M(A) := < \{\lambda_a, \varrho_a \mid a \in A\} > \subset \mathrm{End}_{\mathbb{Z}}(A)$$

- A is $M(A)$-module, generated by 1_A,
- $\mathrm{End}_{M(A)}(A) = Z(A)$ (center), $\quad M(A) \subset \mathrm{End}(A_{Z(A)})$,
- for every ideal $I \subset A$, $\mathrm{Hom}_{M(A)}(A, I) = I \cap Z(A)$,

Multiplication algebra $M(A)$

$$\lambda_a : A \to A,\ x \mapsto ax; \quad \varrho_a : A \to A,\ x \mapsto xa;$$

$$M(A) := < \{\lambda_a, \varrho_a \,|\, a \in A\} > \subset \mathrm{End}_{\mathbb{Z}}(A)$$

- A is $M(A)$-module, generated by 1_A,
- $\mathrm{End}_{M(A)}(A) = Z(A)$ (center), $\quad M(A) \subset \mathrm{End}(A_{Z(A)})$,
- for every ideal $I \subset A$, $\mathrm{Hom}_{M(A)}(A, I) = I \cap Z(A)$,
- ${}_{M(A)}A$ self-projective $\Rightarrow Z(A/I) = Z(A)/(I \cap Z(A))$,

Multiplication algebra $M(A)$

$$\lambda_a : A \to A,\ x \mapsto ax; \quad \varrho_a : A \to A,\ x \mapsto xa;$$

$$M(A) := < \{\lambda_a, \varrho_a \,|\, a \in A\} > \subset \mathrm{End}_{\mathbb{Z}}(A)$$

- A is $M(A)$-module, generated by 1_A,
- $\mathrm{End}_{M(A)}(A) = Z(A)$ (center), $\quad M(A) \subset \mathrm{End}(A_{Z(A)})$,
- for every ideal $I \subset A$, $\mathrm{Hom}_{M(A)}(A, I) = I \cap Z(A)$,
- ${}_{M(A)}A$ self-projective $\Rightarrow Z(A/I) = Z(A)/(I \cap Z(A))$,
- $\sigma[A] = \{\ M(A)$-modules subgenerated by $A\ \} \subset {}_{M(A)}\mathbb{M}$

Category theory

Multiplication algebra $M(A)$

$$\lambda_a : A \to A,\ x \mapsto ax; \quad \varrho_a : A \to A,\ x \mapsto xa;$$

$$M(A) := < \{\lambda_a, \varrho_a \,|\, a \in A\} > \subset \mathrm{End}_{\mathbb{Z}}(A)$$

- A is $M(A)$-module, generated by 1_A,
- $\mathrm{End}_{M(A)}(A) = Z(A)$ (center), $\quad M(A) \subset \mathrm{End}(A_{Z(A)})$,
- for every ideal $I \subset A$, $\mathrm{Hom}_{M(A)}(A, I) = I \cap Z(A)$,
- ${}_{M(A)}A$ self-projective $\Rightarrow Z(A/I) = Z(A)/(I \cap Z(A))$,
- $\sigma[A] = \{\ M(A)$-modules subgenerated by $A\ \} \subset {}_{M(A)}\mathbb{M}$

$$\mathrm{Hom}_{M(A)}(A, -) : \quad {}_{M(A)}\mathbb{M} \to {}_{Z(A)}\mathbb{M},$$
$$A \otimes_{Z(A)} - : \quad {}_{Z(A)}\mathbb{M} \to {}_{M(A)}\mathbb{M}$$

Azumaya rings

Goro Azumaya, 1920 -2010

On maximally central algebra, 1951 (Nagoya J.)

Separable rings, 1980 (J. Algebra)

Azumaya rings

Goro Azumaya, 1920 -2010

On maximally central algebra, 1951 (Nagoya J.)

Separable rings, 1980 (J. Algebra)

Azumaya algebras

$$A \otimes_{Z(A)} - : {}_{Z(A)}\mathbb{M} \to {}_{M(A)}\mathbb{M} \quad \text{is equivalence of categories}$$

Azumaya rings

Goro Azumaya, 1920 -2010

On maximally central algebra, 1951 (Nagoya J.)

Separable rings, 1980 (J. Algebra)

Azumaya algebras

$A \otimes_{Z(A)} - : {}_{Z(A)}\mathbb{M} \to {}_{M(A)}\mathbb{M}$ is equivalence of categories

Azumaya rings

$A \otimes_{Z(A)} - : {}_{Z(A)}\mathbb{M} \to \sigma[A]$ is equivalence of categories

Azumaya rings

Goro Azumaya, 1920 -2010
On maximally central algebra, 1951 (Nagoya J.)
Separable rings, 1980 (J. Algebra)

Azumaya algebras

$A \otimes_{Z(A)} - : {}_{Z(A)}\mathbb{M} \to {}_{M(A)}\mathbb{M}$ is equivalence of categories

Azumaya rings

$A \otimes_{Z(A)} - : {}_{Z(A)}\mathbb{M} \to \sigma[A]$ is equivalence of categories

$A_{Z(A)}$ finitely generated: A Azumaya ring $\Leftrightarrow$ A Azumaya algebra

Azumaya rings

Goro Azumaya, 1920 -2010
On maximally central algebra, 1951 (Nagoya J.)
Separable rings, 1980 (J. Algebra)

Azumaya algebras

$$A \otimes_{Z(A)} - : {}_{Z(A)}\mathbb{M} \to {}_{M(A)}\mathbb{M} \quad \text{is equivalence of categories}$$

Azumaya rings

$$A \otimes_{Z(A)} - : {}_{Z(A)}\mathbb{M} \to \sigma[A] \quad \text{is equivalence of categories}$$

$A_{Z(A)}$ finitely generated: A Azumaya ring $\Leftrightarrow$ A Azumaya algebra

Artin (1969), Delale (1974), Wisbauer (1977), Burkholder (1986)

Semiprime algebras - $\sigma[A]$ Grothendieck catgeory

Central closure of semiprime algebra A (Martindale)

- $\widehat{A}$ injective hull of A in $\sigma[A]$;

Semiprime algebras - $\sigma[A]$ Grothendieck catgeory

Central closure of semiprime algebra A (Martindale)

- $\widehat{A}$ injective hull of A in $\sigma[A]$;
- A-generated: $\widehat{A} = A\,\mathrm{Hom}_{M(A)}(A, \widehat{A})$;

Semiprime algebras - $\sigma[A]$ Grothendieck catgeory

Central closure of semiprime algebra A (Martindale)

- $\widehat{A}$ injective hull of A in $\sigma[A]$;
- A-generated: $\widehat{A} = A \operatorname{Hom}_{M(A)}(A, \widehat{A})$;
- A semiprime $\Rightarrow \operatorname{Hom}_{M(A)}(A, \widehat{A}) = \operatorname{End}_{M(A)}(\widehat{A})$ (extended centroid, commutative, regular);

Semiprime algebras - $\sigma[A]$ Grothendieck catgeory

Central closure of semiprime algebra A (Martindale)

- $\widehat{A}$ injective hull of A in $\sigma[A]$;
- A-generated: $\widehat{A} = A\,\mathrm{Hom}_{M(A)}(A, \widehat{A})$;
- A semiprime $\Rightarrow \mathrm{Hom}_{M(A)}(A, \widehat{A}) = \mathrm{End}_{M(A)}(\widehat{A})$ (extended centroid, commutative, regular);
- $\widehat{A} = A\,\mathrm{End}_{M(A)}(\widehat{A})$ has ring structure (central closure);

Semiprime algebras - $\sigma[A]$ Grothendieck catgeory

Central closure of semiprime algebra A (Martindale)

- $\widehat{A}$ injective hull of A in $\sigma[A]$;
- A-generated: $\widehat{A} = A\,\mathrm{Hom}_{M(A)}(A, \widehat{A})$;
- A semiprime $\Rightarrow \mathrm{Hom}_{M(A)}(A, \widehat{A}) = \mathrm{End}_{M(A)}(\widehat{A})$ (extended centroid, commutative, regular);
- $\widehat{A} = A\,\mathrm{End}_{M(A)}(\widehat{A})$ has ring structure (central closure);
- if A is prime, then $\mathrm{End}_{M(A)}(\widehat{A})$ is a field;

Semiprime algebras - $\sigma[A]$ Grothendieck catgeory

Central closure of semiprime algebra A (Martindale)

- $\widehat{A}$ injective hull of A in $\sigma[A]$;
- A-generated: $\widehat{A} = A\,\mathrm{Hom}_{M(A)}(A, \widehat{A})$;
- A semiprime $\Rightarrow$ $\mathrm{Hom}_{M(A)}(A, \widehat{A}) = \mathrm{End}_{M(A)}(\widehat{A})$ (extended centroid, commutative, regular);
- $\widehat{A} = A\,\mathrm{End}_{M(A)}(\widehat{A})$ has ring structure (central closure);
- if A is prime, then $\mathrm{End}_{M(A)}(\widehat{A})$ is a field;
- if A is prime, for ideals $I \subset A$, $I \cap Z(A) \neq 0$, then $\widehat{A}$ is simple;

Semiprime algebras - $\sigma[A]$ Grothendieck catgeory

Central closure of semiprime algebra A (Martindale)

- $\widehat{A}$ injective hull of A in $\sigma[A]$;
- A-generated: $\widehat{A} = A\,\mathrm{Hom}_{M(A)}(A, \widehat{A})$;
- A semiprime $\Rightarrow \mathrm{Hom}_{M(A)}(A, \widehat{A}) = \mathrm{End}_{M(A)}(\widehat{A})$ (extended centroid, commutative, regular);
- $\widehat{A} = A\,\mathrm{End}_{M(A)}(\widehat{A})$ has ring structure (central closure);
- if A is prime, then $\mathrm{End}_{M(A)}(\widehat{A})$ is a field;
- if A is prime, for ideals $I \subset A$, $I \cap Z(A) \neq 0$, then $\widehat{A}$ is simple;
- for $A = \mathbb{Z}$, $\widehat{\mathbb{Z}} = \mathbb{Q}$.

Separability

Ernst Steinitz, 1871 - 1928, Berlin

Separability

Ernst Steinitz, 1871 - 1928, Berlin

Algebraische Theorie der Körper, 1910

Separability

Ernst Steinitz, 1871 - 1928, Berlin

Algebraische Theorie der Körper, 1910

$L : K$ separable field extension

For $a \in L$, minimal polynomial $f \in K[X]$ has no multiple roots

Separability

Ernst Steinitz, 1871 - 1928, Berlin

Algebraische Theorie der Körper, 1910

$L : K$ separable field extension

For $a \in L$, minimal polynomial $f \in K[X]$ has no multiple roots

Hassler Whitney, 1907 - 1989, Princeton

Tensor products of Abelian groups, 1938

Separability

Ernst Steinitz, 1871 - 1928, Berlin

Algebraische Theorie der Körper, 1910

$L : K$ separable field extension

For $a \in L$, minimal polynomial $f \in K[X]$ has no multiple roots

Hassler Whitney, 1907 - 1989, Princeton

Tensor products of Abelian groups, 1938

$L : K$ field extension: Tensor functor

$$L \otimes_K - : \mathbb{M}_K \to \mathbb{M}_K,\ M \mapsto L \otimes_K M$$

Functor $L \otimes_K - : \mathbb{M}_K \to \mathbb{M}_K$

Product and unit

$$m : L \otimes_K L \to L, \quad \iota : K \to L$$

Functor $L \otimes_K - : \mathbb{M}_K \to \mathbb{M}_K$

Product and unit

$$m : L \otimes_K L \to L, \quad \iota : K \to L$$

L-module

$$L \otimes_K M \xrightarrow{\varrho} M, \quad M \xrightarrow{\iota \otimes M} L \otimes M \xrightarrow{\varrho} M = 1_M$$

Functor $L \otimes_K - : \mathbb{M}_K \to \mathbb{M}_K$

Product and unit

$$m : L \otimes_K L \to L, \quad \iota : K \to L$$

***L*-module**

$$L \otimes_K M \xrightarrow{\varrho} M, \quad M \xrightarrow{\iota \otimes M} L \otimes M \xrightarrow{\varrho} M = 1_M$$

Free and forgetful functor

$$\phi_L : \mathbb{M}_K \to \mathbb{M}_L, \quad M \mapsto (L \otimes_K M, m \otimes M)$$

Functor $L \otimes_K - : \mathbb{M}_K \to \mathbb{M}_K$

Product and unit

$$m : L \otimes_K L \to L, \quad \iota : K \to L$$

L-module

$$L \otimes_K M \xrightarrow{\varrho} M, \quad M \xrightarrow{\iota \otimes M} L \otimes M \xrightarrow{\varrho} M = 1_M$$

Free and forgetful functor

$$\phi_L : \mathbb{M}_K \to \mathbb{M}_L, \quad M \mapsto (L \otimes_K M, m \otimes M)$$
$$U_L : \mathbb{M}_L \to \mathbb{M}_K, \quad (M, \varrho) \mapsto M$$

Functor $L \otimes_K - : \mathbb{M}_K \to \mathbb{M}_K$

Product and unit

$$m : L \otimes_K L \to L, \quad \iota : K \to L$$

L-module

$$L \otimes_K M \xrightarrow{\varrho} M, \quad M \xrightarrow{\iota \otimes M} L \otimes M \xrightarrow{\varrho} M = 1_M$$

Free and forgetful functor

$$\phi_L : \mathbb{M}_K \to \mathbb{M}_L, \quad M \mapsto (L \otimes_K M, m \otimes M)$$
$$U_L : \mathbb{M}_L \to \mathbb{M}_K, \quad (M, \varrho) \mapsto M$$

Adjoint pair

$$\mathrm{Hom}_L(L \otimes_K M, N) \simeq \mathrm{Hom}_K(M, U_L(N))$$

Separability $L : K$ finite

Equivalent – $L : K$ finite

- $L : K$ is a separable field extension;
- for any field extension $Q : K$, $\mathrm{Nil}(L \otimes_K Q) = 0$;
- $\mathrm{Nil}(L \otimes_K L) = 0$;

Separability $L : K$ finite

Equivalent – $L : K$ finite

- $L : K$ is a separable field extension;
- for any field extension $Q : K$, $\mathrm{Nil}(L \otimes_K Q) = 0$;
- $\mathrm{Nil}(L \otimes_K L) = 0$;
- L is projective as $L \otimes_K L$-module.

Separability $L : K$ finite

Equivalent – $L : K$ finite

- $L : K$ is a separable field extension;
- for any field extension $Q : K$, $\mathrm{Nil}(L \otimes_K Q) = 0$;
- $\mathrm{Nil}(L \otimes_K L) = 0$;
- L is projective as $L \otimes_K L$-module.

Trace $tr : L \to K$; $\lambda_a : L \to L$, $x \mapsto ax$

$$tr : L \xrightarrow{\lambda_a} \mathrm{End}_K(L) \xrightarrow{Tr} K$$

Separability $L : K$ finite

Equivalent – $L : K$ finite

- $L : K$ is a separable field extension;
- for any field extension $Q : K$, $\mathrm{Nil}(L \otimes_K Q) = 0$;
- $\mathrm{Nil}(L \otimes_K L) = 0$;
- L is projective as $L \otimes_K L$-module.

Trace $tr : L \to K$; $\lambda_a : L \to L$, $x \mapsto ax$

$$tr : L \xrightarrow{\lambda_a} \mathrm{End}_K(L) \xrightarrow{Tr} K$$

- $tr : L \to K$ is nondegenerate;
- $\psi : L \to L^*$, $a \mapsto tr(a-)$, is an isomorphism (L-linear);
- $L \otimes_K - \simeq \mathrm{Hom}_K(L, -) : \mathbb{M}_K \to \mathbb{M}_L$.

Separability – $L : K$ finite

Dual of algebra (coalgebra) – apply $(-)^* = \mathrm{Hom}(-, K)$ to m and ι

$$L^* \xrightarrow{m^*} L^* \otimes_K L^*, \quad L^* \xrightarrow{\iota^*} K,$$

$L : K$ separable: apply $\psi : L \to L^*$

$$\delta : L \to L \otimes_K L, \quad \varepsilon : L \to K$$

Separability – $L : K$ finite

Dual of algebra (coalgebra) – apply $(-)^* = \mathrm{Hom}(-, K)$ to m and ι

$$L^* \xrightarrow{m^*} L^* \otimes_K L^*, \quad L^* \xrightarrow{\iota^*} K,$$

$L : K$ separable: apply $\psi : L \to L^*$

$$\delta : L \to L \otimes_K L, \quad \varepsilon : L \to K$$

Frobenius conditions

$$\begin{array}{ccc} L \otimes L & \xrightarrow{m} & L \\ {\scriptstyle \delta\otimes L}\downarrow & & \downarrow{\scriptstyle \delta} \\ L \otimes L \otimes L & \xrightarrow{L\otimes m} & L \otimes L, \end{array}$$

Separability – $L : K$ finite

Dual of algebra (coalgebra) – apply $(-)^* = \mathrm{Hom}(-, K)$ to m and ι

$$L^* \xrightarrow{m^*} L^* \otimes_K L^*, \quad L^* \xrightarrow{\iota^*} K,$$

$L : K$ separable: apply $\psi : L \to L^*$

$$\delta : L \to L \otimes_K L, \quad \varepsilon : L \to K$$

Frobenius conditions

$$\begin{CD} L \otimes L @>{m}>> L \\ @V{\delta \otimes L}VV @VV{\delta}V \\ L \otimes L \otimes L @>{L \otimes m}>> L \otimes L \end{CD}, \qquad \begin{CD} L \otimes L @>{m}>> L \\ @V{L \otimes \delta}VV @VV{\delta}V \\ L \otimes L \otimes L @>{m \otimes L}>> L \otimes L \end{CD}$$

Separability – $L : K$ finite

Dual of algebra (coalgebra) – apply $(-)^* = \mathrm{Hom}(-, K)$ to m and ι

$$L^* \xrightarrow{m^*} L^* \otimes_K L^*, \quad L^* \xrightarrow{\iota^*} K,$$

$L : K$ separable: apply $\psi : L \to L^*$

$$\delta : L \to L \otimes_K L, \quad \varepsilon : L \to K$$

Frobenius conditions

$$\begin{array}{ccc} L \otimes L & \xrightarrow{m} & L \\ {\scriptstyle \delta \otimes L}\downarrow & & \downarrow{\scriptstyle \delta} \\ L \otimes L \otimes L & \xrightarrow{L \otimes m} & L \otimes L \end{array}, \qquad \begin{array}{ccc} L \otimes L & \xrightarrow{m} & L \\ {\scriptstyle L \otimes \delta}\downarrow & & \downarrow{\scriptstyle \delta} \\ L \otimes L \otimes L & \xrightarrow{m \otimes L} & L \otimes L \end{array}$$

m left L-comodule morphism | δ is left L-module morphism

Separability – $L : K$ finite

Dual of algebra (coalgebra) – apply $(-)^* = \mathrm{Hom}(-, K)$ to m and ι

$$L^* \xrightarrow{m^*} L^* \otimes_K L^*, \quad L^* \xrightarrow{\iota^*} K,$$

$L : K$ separable: apply $\psi : L \to L^*$

$$\delta : L \to L \otimes_K L, \quad \varepsilon : L \to K$$

Frobenius conditions

$$\begin{array}{ccc} L \otimes L & \xrightarrow{m} & L \\ {\scriptstyle \delta\otimes L}\downarrow & & \downarrow{\scriptstyle \delta} \\ L \otimes L \otimes L & \xrightarrow{L\otimes m} & L \otimes L \end{array}, \qquad \begin{array}{ccc} L \otimes L & \xrightarrow{m} & L \\ {\scriptstyle L\otimes\delta}\downarrow & & \downarrow{\scriptstyle \delta} \\ L \otimes L \otimes L & \xrightarrow{m\otimes L} & L \otimes L \end{array}$$

m left L-comodule morphism	δ is left L-module morphism
δ is right L-module morphism	m is right L-comodule morphism.

Separability $L : K$ finite

- (L, m, δ) satisfy Frobenius condition and $m \cdot \delta = 1_L$

Separability $L : K$ finite

- (L, m, δ) satisfy Frobenius condition and $m \cdot \delta = 1_L$

L-module $\varrho : L \otimes_K M \to M$ becomes comodule (L-linear)

$$\omega : M \xrightarrow{\iota \otimes M} L \otimes M \xrightarrow{\delta \otimes M} L \otimes L \otimes M \xrightarrow{L \otimes \varrho} L \otimes M$$

Separability $L : K$ finite

- (L, m, δ) satisfy Frobenius condition and $m \cdot \delta = 1_L$

L-module $\varrho : L \otimes_K M \to M$ becomes comodule (L-linear)

$$\omega : M \xrightarrow{\iota \otimes M} L \otimes M \xrightarrow{\delta \otimes M} L \otimes L \otimes M \xrightarrow{L \otimes \varrho} L \otimes M$$

Forgetful functor $U_L : \mathbb{M}_L \to \mathbb{M}_K$, $M, N \in \mathbb{M}_L$

$\mathrm{Hom}_L(M, N) \to \mathrm{Hom}_K(U_L M, U_L N)$ splits by

Separability $L : K$ finite

- (L, m, δ) satisfy Frobenius condition and $m \cdot \delta = 1_L$

L-module $\varrho : L \otimes_K M \to M$ becomes comodule (L-linear)

$$\omega : M \xrightarrow{\iota \otimes M} L \otimes M \xrightarrow{\delta \otimes M} L \otimes L \otimes M \xrightarrow{L \otimes \varrho} L \otimes M$$

Forgetful functor $U_L : \mathbb{M}_L \to \mathbb{M}_K$, $M, N \in \mathbb{M}_L$

$\mathrm{Hom}_L(M, N) \to \mathrm{Hom}_K(U_L M, U_L N)$ splits by

$\mathrm{Hom}_K(M, N) \to \mathrm{Hom}_L(M, N)$,

Separability $L : K$ finite

- (L, m, δ) satisfy Frobenius condition and $m \cdot \delta = 1_L$

L-module $\varrho : L \otimes_K M \to M$ becomes comodule (L-linear)

$$\omega : M \xrightarrow{\iota \otimes M} L \otimes M \xrightarrow{\delta \otimes M} L \otimes L \otimes M \xrightarrow{L \otimes \varrho} L \otimes M$$

Forgetful functor $U_L : \mathbb{M}_L \to \mathbb{M}_K$, $M, N \in \mathbb{M}_L$

$\mathrm{Hom}_L(M, N) \to \mathrm{Hom}_K(U_L M, U_L N)$ splits by

$\mathrm{Hom}_K(M, N) \to \mathrm{Hom}_L(M, N)$,

$$M \xrightarrow{f} N \quad \mapsto \quad M \xrightarrow{\omega} L \otimes_K M \xrightarrow{L \otimes f} L \otimes_K N \xrightarrow{\varrho'} N$$

Separability $L : K$ finite

- (L, m, δ) satisfy Frobenius condition and $m \cdot \delta = 1_L$

L-module $\varrho : L \otimes_K M \to M$ becomes comodule (L-linear)

$$\omega : M \xrightarrow{\iota \otimes M} L \otimes M \xrightarrow{\delta \otimes M} L \otimes L \otimes M \xrightarrow{L \otimes \varrho} L \otimes M$$

Forgetful functor $U_L : \mathbb{M}_L \to \mathbb{M}_K$, $M, N \in \mathbb{M}_L$

$\mathrm{Hom}_L(M, N) \to \mathrm{Hom}_K(U_L M, U_L N)$ splits by

$\mathrm{Hom}_K(M, N) \to \mathrm{Hom}_L(M, N)$,

$$M \xrightarrow{f} N \quad \mapsto \quad M \xrightarrow{\omega} L \otimes_K M \xrightarrow{L \otimes f} L \otimes_K N \xrightarrow{\varrho'} N$$

- $U_L : \mathbb{M}_L \to \mathbb{M}_K$ is separable functor (to be defined)

R-algebras

Maurice Auslander, Oscar Goldman,
The Brauer group of a commutative ring, 1960

R-algebras

Maurice Auslander, Oscar Goldman,
The Brauer group of a commutative ring, 1960

Separable algebras

(a) A is projective as an $A \otimes_R A^o$-module;

(b) $U_L : {}_A\mathbb{M} \to \mathbb{M}_R$ is a separable functor;

(c) A is separable over $C(A)$ and $C(A)$ is separable over R.

R-algebras

Maurice Auslander, Oscar Goldman,
The Brauer group of a commutative ring, 1960

Separable algebras

(a) A is projective as an $A \otimes_R A^o$-module;

(b) $U_L : {}_A\mathbb{M} \to \mathbb{M}_R$ is a separable functor;

(c) A is separable over $C(A)$ and $C(A)$ is separable over R.

Frobenius algebras (A, m, e)

(a) $A \simeq \mathrm{Hom}_R(A, R)$ as A-modules;

(b) $A \otimes_R - \simeq \mathrm{Hom}_R(A, -) : \mathbb{M}_R \to {}_A\mathbb{M}$;

(c) (A, δ, ε) is a coalgebra, Frobenius condition for (m, δ).

R-algebras

Maurice Auslander, Oscar Goldman,
The Brauer group of a commutative ring, 1960

Separable algebras

(a) A is projective as an $A \otimes_R A^o$-module;
(b) $U_L : {}_A\mathbb{M} \to \mathbb{M}_R$ is a separable functor;
(c) A is separable over $C(A)$ and $C(A)$ is separable over R.

Frobenius algebras (A, m, e)

(a) $A \simeq \mathrm{Hom}_R(A, R)$ as A-modules;
(b) $A \otimes_R - \simeq \mathrm{Hom}_R(A, -) : \mathbb{M}_R \to {}_A\mathbb{M}$;
(c) (A, δ, ε) is a coalgebra, Frobenius condition for (m, δ).

Every module (M, ϱ) is comodule

$$\omega : M \xrightarrow{\iota \otimes M} A \otimes M \xrightarrow{\delta \otimes M} A \otimes A \otimes M \xrightarrow{A \otimes \varrho} A \otimes M$$

Various algebras

Algebra (A, m), coalgebra (A, δ), Frobenius condition

$$\begin{array}{ccc} A \otimes A & \xrightarrow{m} & A \\ {\scriptstyle I\otimes\delta}\downarrow & & \downarrow{\scriptstyle \delta} \\ A \otimes A \otimes A & \xrightarrow{m\otimes I} & A \otimes A \end{array} \qquad \begin{array}{ccc} A \otimes A & \xrightarrow{m} & A \\ {\scriptstyle \delta\otimes I}\downarrow & & \downarrow{\scriptstyle \delta} \\ A \otimes A \otimes A & \xrightarrow{I\otimes m} & A \otimes A \end{array}$$

Various algebras

Algebra (A, m), coalgebra (A, δ), Frobenius condition

$$
\begin{array}{ccc}
A \otimes A & \xrightarrow{m} & A \\
{\scriptstyle I\otimes\delta}\downarrow & & \downarrow{\scriptstyle \delta} \\
A \otimes A \otimes A & \xrightarrow{m\otimes I} & A \otimes A
\end{array}
\qquad
\begin{array}{ccc}
A \otimes A & \xrightarrow{m} & A \\
{\scriptstyle \delta\otimes I}\downarrow & & \downarrow{\scriptstyle \delta} \\
A \otimes A \otimes A & \xrightarrow{I\otimes m} & A \otimes A
\end{array}
$$

Frobenius algebra $(A, m, e; \delta, \varepsilon)$ – Frobenius modules

equivalence $\quad A \otimes_R - : {}_A\mathbb{M} \to {}_A^A\mathbb{M}$

Various algebras

Algebra (A, m), coalgebra (A, δ), Frobenius condition

$$\begin{array}{ccc} A \otimes A & \xrightarrow{m} & A \\ {\scriptstyle I\otimes\delta}\downarrow & & \downarrow{\scriptstyle \delta} \\ A \otimes A \otimes A & \xrightarrow{m\otimes I} & A \otimes A \end{array} \qquad \begin{array}{ccc} A \otimes A & \xrightarrow{m} & A \\ {\scriptstyle \delta\otimes I}\downarrow & & \downarrow{\scriptstyle \delta} \\ A \otimes A \otimes A & \xrightarrow{I\otimes m} & A \otimes A \end{array}$$

Frobenius algebra $(A, m, e; \delta, \varepsilon)$ – Frobenius modules

equivalence $\quad A \otimes_R - : {}_A\mathbb{M} \to {}^A_A\mathbb{M}$

Separable algebra $(A, m, e; \delta)$, $\quad m \circ \delta = I$

Various algebras

Algebra (A, m), coalgebra (A, δ), Frobenius condition

$$\begin{array}{ccc} A \otimes A & \xrightarrow{m} & A \\ {\scriptstyle I\otimes\delta}\downarrow & & \downarrow{\scriptstyle \delta} \\ A \otimes A \otimes A & \xrightarrow{m\otimes I} & A \otimes A \end{array} \qquad \begin{array}{ccc} A \otimes A & \xrightarrow{m} & A \\ {\scriptstyle \delta\otimes I}\downarrow & & \downarrow{\scriptstyle \delta} \\ A \otimes A \otimes A & \xrightarrow{I\otimes m} & A \otimes A \end{array}$$

Frobenius algebra $(A, m, e; \delta, \varepsilon)$ – Frobenius modules

equivalence $A \otimes_R - : {}_A\mathbb{M} \to {}_A^A\mathbb{M}$

Separable algebra $(A, m, e; \delta)$, $m \circ \delta = I$

Azumaya algebra $(A, m, e; \delta)$, $m \circ \delta = I$ – bimodules

separable and central, equivalence $A \otimes_R - : \mathbb{M}_R \to {}_A\mathbb{M}_A$

Separable functors $\mathrm{Hom}_L(M, N) \to \mathrm{Hom}_K(U_L(M), U_L(N))$

Separable functors $\mathrm{Hom}_L(M,N) \to \mathrm{Hom}_K(U_L(M), U_L(N))$

C. Năstăsescu, M. Van den Bergh,
F. Van Oystaeyen, 1989 (J. Algebra)
Separable functors applied to graded rings

Separable functors $\mathrm{Hom}_L(M, N) \rightarrowtail \mathrm{Hom}_K(U_L(M), U_L(N))$

C. Năstăsescu, M. Van den Bergh,
F. Van Oystaeyen, 1989 (J. Algebra)
Separable functors applied to graded rings

$F : \mathbb{A} \to \mathbb{B}$ separable functor

$$\Phi^F : \mathrm{Hom}_{\mathbb{A}}(A, A') \to \mathrm{Hom}_{\mathbb{B}}(F(A), F(A')) \text{ nat. split mono}$$

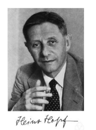
C. Năstăsescu, M. Van den Bergh,
F. Van Oystaeyen, 1989 (J. Algebra)
Separable functors applied to graded rings

$F : \mathbb{A} \to \mathbb{B}$ separable functor

$$\Phi^F : \mathrm{Hom}_{\mathbb{A}}(A, A') \to \mathrm{Hom}_{\mathbb{B}}(F(A), F(A')) \text{ nat. split mono}$$

$$\Psi : \mathrm{Hom}_{\mathbb{B}}(F(A), F(A')) \to \mathrm{Hom}_{\mathbb{A}}(A, A'), \quad \Psi \circ \Phi^F = I$$

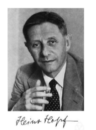
Hopf algebras



Heinz Hopf, 1894 -1971, Zürich

Über die Topologie der Gruppen-Mannigfaltigkeiten und ihre Verallgemeinerungen, 1941

Hopf algebras



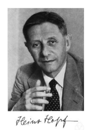
Heinz Hopf, 1894 -1971, Zürich

Über die Topologie der Gruppen-Mannigfaltigkeiten und ihre Verallgemeinerungen, 1941

Bialgebra: algebra (A, m, ι), coalgebra (A, δ, ε)

$\delta : A \to A \otimes_K A, \quad \varepsilon : A \to K$ are algebra morphisms

Hopf algebras



Heinz Hopf, 1894 -1971, Zürich

Über die Topologie der Gruppen-Mannigfaltigkeiten und ihre Verallgemeinerungen, 1941

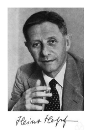
Bialgebra: algebra (A, m, ι), coalgebra (A, δ, ε)

$$\delta : A \to A \otimes_K A, \quad \varepsilon : A \to K \text{ are algebra morphisms}$$

Bimodules (Hopf modules) ${}^A_A\mathbb{M}$

$$\begin{array}{ccc} A\otimes M \xrightarrow{\;\varrho\;} & M & \xrightarrow{\;\omega\;} A\otimes M \\ \downarrow{\scriptstyle \delta\otimes\omega} & & \uparrow{\scriptstyle m\otimes\varrho} \\ A\otimes A\otimes A\otimes M & \xrightarrow{A\otimes\tau\otimes M} & A\otimes A\otimes A\otimes M \end{array}$$

Hopf algebras



Heinz Hopf, 1894 -1971, Zürich

Über die Topologie der Gruppen-Mannigfaltigkeiten und ihre Verallgemeinerungen, 1941

Bialgebra: algebra (A, m, ι), coalgebra (A, δ, ε)

$$\delta : A \to A \otimes_K A, \quad \varepsilon : A \to K \text{ are algebra morphisms}$$

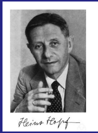
Bimodules (Hopf modules) $\,{}^{A}_{A}\mathbb{M}$

$$\begin{array}{ccccc}
A \otimes M & \xrightarrow{\varrho} & M & \xrightarrow{\omega} & A \otimes M \\
\delta\otimes\omega \downarrow & & & & \uparrow m\otimes\varrho \\
A \otimes A \otimes A \otimes M & & \xrightarrow{A\otimes\tau\otimes M} & & A \otimes A \otimes A \otimes M
\end{array}$$

Hopf algebra

$$A \otimes A \xrightarrow{\delta\otimes A} A \otimes A \otimes A \xrightarrow{A\otimes m} A \otimes A = 1_{A\otimes A}$$

Hopf algebras



Heinz Hopf, 1894 -1971, Zürich

Über die Topologie der Gruppen-Mannigfaltigkeiten und ihre Verallgemeinerungen, 1941

Bialgebra: algebra (A, m, ι), coalgebra (A, δ, ε)

$$\delta : A \to A \otimes_K A, \quad \varepsilon : A \to K \text{ are algebra morphisms}$$

Bimodules (Hopf modules) ${}^A_A\mathbb{M}$

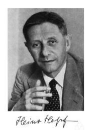
$$\begin{array}{ccccc} A \otimes M & \xrightarrow{\varrho} & M & \xrightarrow{\omega} & A \otimes M \\ \downarrow{\scriptstyle \delta\otimes\omega} & & & & \uparrow{\scriptstyle m\otimes\varrho} \\ A \otimes A \otimes A \otimes M & & \xrightarrow{A\otimes\tau\otimes M} & & A \otimes A \otimes A \otimes M \end{array}$$

Hopf algebra

$$A \otimes A \xrightarrow{\delta\otimes A} A \otimes A \otimes A \xrightarrow{A\otimes m} A \otimes A = 1_{A\otimes A}$$

equivalence $\quad A \otimes_R - : \mathbb{M}_R \to {}^A_A\mathbb{M}$

Hopf algebras



Heinz Hopf, 1894 -1971, Zürich

Über die Topologie der Gruppen-Mannigfaltigkeiten und ihre Verallgemeinerungen, 1941

Bialgebra: algebra (A, m, ι), coalgebra (A, δ, ε)

$\delta : A \to A \otimes_K A, \quad \varepsilon : A \to K$ are algebra morphisms

Bimodules (Hopf modules) $\quad {}^A_A\mathbb{M}$

$$\begin{array}{ccccc} A \otimes M & \xrightarrow{\varrho} & M & \xrightarrow{\omega} & A \otimes M \\ {\scriptstyle \delta\otimes\omega}\downarrow & & & & \uparrow{\scriptstyle m\otimes\varrho} \\ A \otimes A \otimes A \otimes M & & \xrightarrow{A\otimes\tau\otimes M} & & A \otimes A \otimes A \otimes M \end{array}$$

Hopf algebra

$$A \otimes A \xrightarrow{\delta\otimes A} A \otimes A \otimes A \xrightarrow{A\otimes m} A \otimes A = 1_{A\otimes A}$$

equivalence $\quad A \otimes_R - : \mathbb{M}_R \to {}^A_A\mathbb{M} \quad$ (antipode $\quad S : A \to A$)

$G = \mathrm{Aut}(L : K)$

Group ring $K[G]$ – algebra and coalgebra – bialgebra

$$m : K[G] \otimes K[G] \to K[G], \quad g \otimes h \mapsto hg,$$

$G = \mathrm{Aut}(L : K)$

Group ring $K[G]$ – algebra and coalgebra – bialgebra

$$m : K[G] \otimes K[G] \to K[G], \quad g \otimes h \mapsto hg,$$

$$\delta : K[G] \to K[G] \otimes K[G], \quad g \mapsto g \otimes g.$$

$G = \mathrm{Aut}(L : K)$

Group ring $K[G]$ – algebra and coalgebra – bialgebra

$$m : K[G] \otimes K[G] \to K[G], \quad g \otimes h \mapsto hg,$$

$$\delta : K[G] \to K[G] \otimes K[G], \quad g \mapsto g \otimes g.$$

Dual group ring $(K[G]^*, m^*, \delta^*)$ – L comodule algebra

L is $K[G]^*$-comodule: $\{g_i, p_i\}_{i \leq n}$ dual basis for $K[G]$

$$\omega : L \to L \otimes_K K[G]^*, \qquad a \mapsto \sum_i g_i(a) \otimes p_i$$

$G = \mathrm{Aut}(L : K)$

Group ring $K[G]$ – algebra and coalgebra – bialgebra

$$m : K[G] \otimes K[G] \to K[G], \quad g \otimes h \mapsto hg,$$

$$\delta : K[G] \to K[G] \otimes K[G], \quad g \mapsto g \otimes g.$$

Dual group ring $(K[G]^*, m^*, \delta^*)$ – L comodule algebra

L is $K[G]^*$-comodule: $\{g_i, p_i\}_{i \leq n}$ dual basis for $K[G]$

$$\omega : L \to L \otimes_K K[G]^*, \qquad a \mapsto \sum_i g_i(a) \otimes p_i$$

Canonical map

$$\beta : L \otimes_K L \to L \otimes_K K[G]^*, \qquad b \otimes a \mapsto \sum_i b g_i(a) \otimes p_i$$

$G = \mathrm{Aut}(L : K)$

Group ring $K[G]$ – algebra and coalgebra – bialgebra

$$m : K[G] \otimes K[G] \to K[G], \quad g \otimes h \mapsto hg,$$
$$\delta : K[G] \to K[G] \otimes K[G], \quad g \mapsto g \otimes g.$$

Dual group ring $(K[G]^*, m^*, \delta^*)$ – L comodule algebra

L is $K[G]^*$-comodule: $\{g_i, p_i\}_{i \le n}$ dual basis for $K[G]$

$$\omega : L \to L \otimes_K K[G]^*, \qquad a \mapsto \sum_i g_i(a) \otimes p_i$$

Canonical map

$$\beta : L \otimes_K L \to L \otimes_K K[G]^*, \qquad b \otimes a \mapsto \sum_i b g_i(a) \otimes p_i$$

$L : K$ separable and normal $\Leftrightarrow$ β is an isomorphism

Thank you !