Bricks over preprojective algebras

Sota Asai, Nagoya University

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Setting

- K: a field.
- Δ: a Dynkin diagram.
- Π: the preprojective K-algebra for Δ.
- $\operatorname{mod}\Pi$: the category of fin. dim. left Π-modules.

Semibricks

Definition

Let $S \in \operatorname{mod} \Pi$.

(1) S: a brick $:\Longleftrightarrow$ $\operatorname{End}_{\Pi}(S)$ is a division K-algebra.
brick $\Pi := \{$all bricks in mod $\Pi\}$.

(2) S: a semibrick $:\Longleftrightarrow$
$S = \bigoplus_{i=1}^{m} S_i$ with each S_i a brick and
$\operatorname{Hom}_{\Pi}(S_i, S_j) = 0$ for $i \neq j$.
sbrick $\Pi := \{$all semibricks in mod $\Pi\}$.

Lattices

Definition

A poset L is called a lattice if L admits the meet $x \wedge y$ and the join $x \vee y$ for $x, y \in L$.
j-irr $L := \{$all join-irreducible elements in $L\}$.

- W: the Coxeter group for Δ (a finite group).
- $\{s_i\}_{i \in \Delta_0}$: the generators of W.
- $\leq \, = \, \leq_{\mathrm{L}}$: the left weak order on W.
- torf $\Pi := \{$torsion-free classes in mod $\Pi\}$.

Lemma

$(W, \leq)$ and (torf $\Pi, \subset)$ are lattices.

Bijections

Let $w = s_{i_1} s_{i_2} \cdots s_{i_l} \in W$ be a reduced expression.

- $I(w) := I_{i_1} I_{i_2} \cdots I_{i_l} \subset \Pi$: an ideal.
- $J(w) := \Pi / I(w)$.

Proposition [Mizuno]

There exists an isomorphism $(W, \leq) \longrightarrow (\mathrm{torf}\,\Pi, \subset)$ of finite lattices given by $w \mapsto \mathrm{Sub}\,J(w)$.

Coxeter groups and semibricks

Proposition [A]

There exist bijections

$$W \longrightarrow \mathsf{sbrick}\,\Pi, \quad \text{j-irr}\,W \longrightarrow \mathsf{brick}\,\Pi$$

given by $w \mapsto S(w) := \mathsf{soc}_{\mathsf{End}_\Pi(J(w))} J(w)$.

- There is a combinatorial result on $J(w)$ for $w \in \text{j-irr}\,W$ [Iyama–Reading–Reiten–Thomas].
- Thus, if $w \in \text{j-irr}\,W$, then one can calculate $S(w) \in \mathsf{brick}\,\Pi$.

Q & A

Question

Let $w \in W$ and $w_1, w_2, \ldots, w_m \in$ j-irr W satisfy $S(w) = \bigoplus_{i=1}^{m} S(w_i) \in$ sbrick Π.
Then, how are w and $w_1, w_2, \ldots, w_m$ related?

Theorem [A]

$w = \bigvee_{i=1}^{m} w_i$ is the canonical join representation.

$w_1, w_2, \ldots, w_m$ are the "minimal generators" of w.

Canonical join representations

Let L be a finite lattice.

Definition [Reading]

Let $x \in L$ and $U \subset \text{j-irr}\, L$ satisfy $x = \bigvee_{u \in U} u$.
$x = \bigvee_{u \in U} u$: a canonical join representation : $\iff$

(a) for any $U' \subsetneq U$, $x \neq \bigvee_{u \in U'} u$;

(b) if $V \subset \text{j-irr}\, L$ satisfies $x = \bigvee_{v \in V} v$ and (a), then $\forall u \in U$, $\exists v \in V$ such that $u \leq v$.

- If a canonical join representation of $x \in L$ exists, then it is unique.
- Every $w \in W$ has a canonical join representation, though it does not hold for general finite lattices.

Conclusion

In my poster...

- similar observations holding
  for general τ-tilting finite algebras;
- an explicit way to get canonical join representations
  in the case $\Delta = \mathbb{A}_n$;
- a description of brick $S(w)$ for $w \in$ j-irr W
  in the case $\Delta = \mathbb{A}_n$.

Thank you very much.