Relative non-commuting graph of a finite ring

Dhiren Kumar Basnet

Tezpur University, India

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University of Yamanashi

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Outlines

Introduction

Definition

Let R be a finite ring. The non-commuting graph of R, denoted by Γ_R, is a simple undirected graph whose vertex set is $R \setminus Z(R)$ and two distinct vertices a and b are adjacent if and only if $ab \neq ba$.

♦ In 2015, Erfanian et al. [3] initiated the study of the non-commuting graph of a finite ring R.

Introduction

Motivated by the works of Tolue and Erfanian [5], we have defined the following graph.

Definition

Let S be a subring of a finite ring R and $C_R(S) = \{r \in R : rs = sr \; \forall \; s \in S\}$. The relative non-commuting graph of the subring S in R, denoted by $\Gamma_{S,R}$, is a simple undirected graph whose vertex set is $R \setminus C_R(S)$ and two distinct vertices a, b are adjacent if and only if a or $b \in S$ and $ab \neq ba$.

- For $S = R, \Gamma_{S,R} = \Gamma_R$, the non-commuting graph of R.
- $\Gamma_{S,R}$ is empty graph if and only if S is commutative.

Let $\mathcal{G}$ be a graph. We write

- $V(\mathcal{G}) :=$ the set of vertices of $\mathcal{G}$,
- $E(\mathcal{G}) :=$ the set of edges of $\mathcal{G}$,
- $diam(\mathcal{G}) := max\{d(x, y) : x, y \in V(\mathcal{G})\}$, where $d(x, y)$ is the length of the shortest path from x to y,
- $girth(\mathcal{G}) :=$ the length of the shortest cycle obtained in $\mathcal{G}$.

Our results

We have obtained the following main results.

Theorem 1

Let S be a non-commutative subring of a finite ring R. Then

1. $\deg(r) = |R| - |C_R(r)|$ if $r \in V(\Gamma_{S,R}) \cap S$.
2. $\deg(r) = |S| - |C_S(r)|$ if $r \in V(\Gamma_{S,R}) \cap (R \setminus S)$.

Theorem 2

Let S be a non-commutative subring of a ring R. If $Z(S) = \{0\}$ then $diam(\Gamma_{S,R}) = 2$ and $girth(\Gamma_{S,R}) = 3$.

Some families of graphs

Star graph: A star graph is a tree on n vertices in which one vertex has degree $n-1$ and the others have degree 1.

Regular graph: A regular graph is a graph where each vertex has the same degree or valency.

Bipartite graph: A bipartite graph is a graph whose vertex set can be partitioned into two disjoint parts in such a way that the two end vertices of every edge lie in different parts.

Complete graph: A complete graph is a graph in which every pair of distinct vertices is adjacent.

Our Result

We have obtained the following result.

Theorem 3

Let S be non-commutative subgroup a finite ring R. Then

1. $\Gamma_{S,R}$ is not a star graph or a bipartite graph.
2. $\Gamma_{S,R}$ is not an n-regular graph for any square free odd positive integer n, where S is a proper subring of R.
3. $\Gamma_{S,R}$ is not a complete graph, where R has unity.

Our result

Let $\mathcal{G}$ be a graph and D a subset of $V(\mathcal{G})$. D is called a dominating set for $\mathcal{G}$ if every vertex in $V(\mathcal{G}) \setminus D$ is adjacent to at least one member of D.

We have obtained the following results.

Theorem 4

Let S be a subring of a ring R and $A \subseteq V(\Gamma_{S,R})$. Then A is a dominating set for $\Gamma_{S,R}$ if and only if $C_R(A) \subseteq A \cup C_R(S)$.

Our result

Theorem 5

Let R be a finite non-commutative ring with unity and S a subring of R. Let $A = \{s_1, s_2, \ldots, s_n\}$ is a generating set for S and $A \cap C_R(S) = \{s_{m+1}, \ldots, s_n\}$ then $D = \{s_1, s_2, \ldots, s_m\} \cup \{s_1 + s_{m+1}, s_2 + s_{m+2}, \ldots, s_1 + s_n\}$ is a dominating set for $\Gamma_{S,R}$.

Our results

In [2], we have defined the relative commuting probability of R relative to a subring S of R as the ratio

$$\Pr(S,R) := \frac{|\{(s,r) \in S \times R : sr = rs\}|}{|S||R|}.$$

We have obtained the following relation between $|E(\Gamma_{S,R})|$ and $\Pr(S,R)$.

Theorem 6

Let S be a subring of a ring R. Then the number of edges of $\Gamma_{S,R}$ is

$$|E(\Gamma_{S,R})| = |S||R|(1 - \Pr(S,R)) - \frac{|S|^2}{2}(1 - \Pr(S)).$$

Our result

We have obtained the following results as consequences of Theorem 6.

Corollary 1

Let S be a non-commutative subring of a ring R and p the smallest prime dividing $|R|$. Then

$$|E(\Gamma_{S,R})| \leq |S|\left(|R| - \frac{3|S|}{16} - p\right) - |Z(R) \cap S|(|R| - p)$$

Our result

Corollary 2

Let S be a non-commutative subring of a ring R. Then

$$|E(\Gamma_{S,R})| \geq -\frac{3|S|^2}{16} + \frac{3|S||R|}{8}.$$

Definition

Buckley et al. [1] introduced the concept of $\mathbb{Z}$-isoclinism between two rings. Motivated by them, we introduce the concept of $\mathbb{Z}$-isoclinism between two pairs of rings.

Definition

Let S_1 and S_2 be two subrings of R_1 and R_2 respectively. A pair of rings (S_1, R_1) is said to be $\mathbb{Z}$-isoclinic to a pair of rings (S_2, R_2) if there exist additive group isomorphisms $\phi : \frac{R_1}{Z(R_1)\cap S_1} \to \frac{R_2}{Z(R_2)\cap S_2}$ such that $\phi\left(\frac{S_1}{Z(R_1)\cap S_1}\right) = \frac{S_2}{Z(R_2)\cap S_2}$; and $\psi : [S_1, R_1] \to [S_2, R_2]$ such that $\psi([u, v]) = [u', v']$ whenever $\phi(u + (Z(R_1) \cap S_1)) = u' + (Z(R_2) \cap S_2), \phi(v + (Z(R_1) \cap S_1)) = v' + (Z(R_2) \cap S_2)$.

Our result

We have obtained the following relation between $\mathbb{Z}$-isoclinism and $\Gamma(S, R)$.

Theorem 7

Let S_1 and S_2 be two subrings of the finite rings R_1 and R_2 respectively. Let the pairs (S_1, R_1) and (S_2, R_2) are $\mathbb{Z}$-isoclinic. Then $\Gamma_{S_1,R_1} \cong \Gamma_{S_2,R_2}$ if $|Z(R_1) \cap S_1| = |Z(R_2) \cap S_2|$ and $|Z(R_1)| = |Z(R_2)|$.

Our result

We conclude with the following corollary.

Corollary 3

Let R be a ring with subrings S and T such that (S, R) is $\mathbb{Z}$-isoclinic to (T, R). Then $\Gamma_S \cong \Gamma_T$ if $|Z(R) \cap S| = |Z(R) \cap T|$.

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THANK YOU