Degenerations of Cohen-Macaulay modules via matrix representations

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Throughout the talk, $(R, \mathfrak{m})$ is a commutative noetherian complete local k-algebra with a residue field $k = \overline{k}$.

- Since R is complete, by Cohen's structure theorem for a complete local rings,

$$\exists S \subset R : \text{a regular local ring s.t. } R \text{ is a module-finite } S\text{-algebra.}$$

- We say that R is Cohen-Macaulay (abbr. CM) if R is free as an S-module. We also say that a finitely generated R-module M is a CM module if M is free as an S-module.

$$\mathrm{CM}(R) = \{M \mid \exists n \in \mathbb{N}, M \cong S^n \text{ as an} S\text{-module.}\}$$

In the rest of the talk, we assume that R is CM.

- Let $M \in \mathrm{CM}(R)$. Since $M \cong S^n$ for some n,

$$\exists\ \mu : R \to \mathrm{End}_S(M) \cong M_n(S); \text{a } k\text{-algebra homomorphism.}$$

- μ is called a **matrix-representation** of M over S.

Example

Let $R = k[[x,y]]/(x^2)$. Then $S = k[[y]] \subset R \cong S \oplus xS$. It is known that R is of countable representation type and isomorphic classes of indecomposable CM modules are the following;

$$R, \quad R/(x), \quad (x, y^n) \quad n \geq 1.$$

Then the matrix representations of these modules are

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (0), \quad \begin{pmatrix} 0 & y^n \\ 0 & 0 \end{pmatrix} \quad n \geq 1.$$

Definition (degeneration (Yoshino, 2004))

Let $V = k[[t]]$. For M, $N \in \mathrm{CM}(R)$. We say that M degenerates to N if there exists a finitely generated $R \otimes_k V$-module Q such that

(1) Q is V-flat.

(2) $Q/tQ \cong N$ as an R-module.

(3) $Q_t \cong M \otimes_k V_t$ as an $R \otimes_k V_t$-module.

Remark

- [Zwara, 2000], [Yoshino, 2004] M degenerates to N if and only if

$$\exists\ 0 \to Z \xrightarrow{\binom{\varphi}{\psi}} M \oplus Z \to N \to 0 \text{ s.t. } \psi^n = 0 \text{ for } n \gg 1.$$

- If $\exists 0 \to L \to M \to N \to 0$, then M degenerates to $L \oplus N$.

History

The degeneration problem of modules has been studied by many authors.

- [Bongartz, 1996] If R is a representation directed k-algebra, $M \leq_{hom} N \Leftrightarrow M \leq_{deg} N$.
- [Zwara, 1999] If R is of finite representation type, $M \leq_{deg} N \Leftrightarrow M \leq_{ext} N$.
- [-, Yoshino, 2013] If R is an even-dimensional hypersurface of finite representation type (A_n), $M \leq_{deg} N \Leftrightarrow M \leq_{ext} N$ on $\mathrm{CM}(R)$.

Today

R is a hypersurface of countable representation type (A_∞), that is $R = k[[x, y]]/(x^2),\ k[[x, y, z]]/(x^2 + y^2)$.

Proposition

Let $M, N \in \mathrm{CM}(R)$. Suppose that M degenerates to N. Let Q be a finitely generated $R \otimes_k V$-module which gives the degeneration. Then Q is **free** as an $S \otimes_k V$-module.

- By the proposition, we can consider the matrix representation of Q over $S \otimes_k V$;

$$\exists\ \xi : R \otimes_k V \to M_n(S \otimes_k V) \quad \text{for some } n \in \mathbb{N}.$$

Corollary

Let $M, N \in \mathrm{CM}(R)$. Then M degenerates to N if and only if there exists the matrix representation ξ over $S \otimes_k V$ such that

$$\xi \otimes_V V/t \cong \nu \quad \text{and} \quad \xi \otimes_V V_t \cong \mu \otimes V_t,$$

where ν and μ are the matrix representation of N and M respectively.

- For matrices μ and ν, we denote by $\mu \cong \nu$ if $\exists P$: invertible matrix s.t. $P^{-1}\mu P = \nu$.

Corollary (Necessary condition for degenerations)

Let $M, N \in \mathrm{CM}(R)$. Suppose that M degenerates to N. Let Q be a finitely generated $R \otimes_k V$-module which gives the degeneration. We denote by μ (resp. ξ) the matrix representation of M (resp. Q). Then we have the following equalities in $S \otimes_k V$:

(1) $\mathrm{tr}(\xi) = \mathrm{tr}(\mu)$,

(2) $\det(\xi) = \det(\mu)$,

(3) $\forall j > 0, \exists l \geq 0$ s.t. $I_j(\xi) = t^l I_j(\mu)$.

- Here, for a matrix μ,
  - $\mathrm{tr}(\mu)$: a trace of μ.
  - $\det(\mu)$: a determinant of μ.
  - $I_j(\mu)$: an ideal generated by j-minors of μ.

Theorem ($\dim R = 1$)

Let $R = k[[x,y]]/(x^2)$. Then $\begin{pmatrix} 0 & y^a \\ 0 & 0 \end{pmatrix}$ degenerates to $\begin{pmatrix} 0 & y^b \\ 0 & 0 \end{pmatrix}$ if and only if $a \leq b$ and $a \equiv b \mod 2$.

Proof.

($\Leftarrow$) We consider $\begin{pmatrix} ty^{\frac{a+b}{2}} & y^b \\ -t^2y^a & -ty^{\frac{a+b}{2}} \end{pmatrix}$ as $\xi \in M_2(S \otimes_k V)$.

($\Rightarrow$) Check the condition on a and b to satisfy the corollary. □

Theorem ($\dim R = 2$)

Let $R = k[[x,y,z]]/(x^2 - yx)$. Then $\begin{pmatrix} 0 & z^a \\ 0 & y \end{pmatrix}$ (resp. $\begin{pmatrix} y & z^a \\ 0 & 0 \end{pmatrix}$) **never** degenerates to $\begin{pmatrix} 0 & z^b \\ 0 & y \end{pmatrix}$ (resp. $\begin{pmatrix} y & z^b \\ 0 & 0 \end{pmatrix}$) for all $a < b$.

Remark

Let R be $k[[x,y,z]]/(x^2 - xy)$. It is also known that R is of countable representation type. The isomorphic classes of indecomposable CM modules are

$$R, \quad (x), \quad (x-y), \quad (x, z^n), \quad (x-y, z^n), \quad n \geq 1.$$

Then the matrix representations of these modules are

$$\begin{pmatrix} y & 1 \\ 0 & 0 \end{pmatrix}, \quad (y), \quad (0), \quad \begin{pmatrix} y & z^n \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & z^n \\ 0 & y \end{pmatrix}, \quad n \geq 1$$

Thank you for your attention.