The defining relations and the Calabi-Yau property of 3-dimensional quadratic AS-regular algebras

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8th October 2017

The 50th Symposium on Ring Theory and Representation Theory @ Yamanashi University

AS-regular algebras

- k: an algebraically closed field with $\operatorname{char} k = 0$,
- A: a connected graded k-algebra finitely generated in degree 1. $(A = \oplus_{i \in \mathbb{N}} A_i\,,\ A_i A_j \subset A_{i+j}\,, A_0 = k.\ )$

Definition ([Artin-Schelter, 1987])

A: *d-dimensional AS-regular algebra* $:\Longleftrightarrow$

(i) $\operatorname{gldim} A = d < \infty$,

(ii) $\operatorname{GKdim} A := \inf\{\alpha \in \mathbb{R} \mid \dim_k(\sum_{i=0}^{n} A_i) \leq n^\alpha, \forall n \gg 0\} < \infty$ (*the Gelfand-Kirillov dimension* of A),

(iii) (*Gorenstein condition*) $\operatorname{Ext}^i_A(k, A) = \begin{cases} k & (i = d), \\ 0 & (i \neq d). \end{cases}$

Geometric algebras

Definition ([Mori, 2006])

- $E \subset \mathbb{P}^{n-1}$: closed subscheme, $\sigma \in \mathrm{Aut}_k E$.
- $A = k\langle x_1, \cdots, x_n \rangle/(R)$, $R \subset k\langle x_1, \ldots, x_n \rangle_2$.

$$\mathcal{V}(R) := \{(p, q) \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \mid f(p, q) = 0, \forall f \in R\}.$$

$A = k\langle x_1, \ldots, x_n \rangle/I$: a quadratic k-algebra.

(1) A satisfies (G1) ($\mathcal{P}(A) = (E, \sigma)$) $:\Longleftrightarrow \exists (E, \sigma)$ s.t. $\mathcal{V}(R) = \{(p, \sigma(p)) \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \mid p \in E\}$.

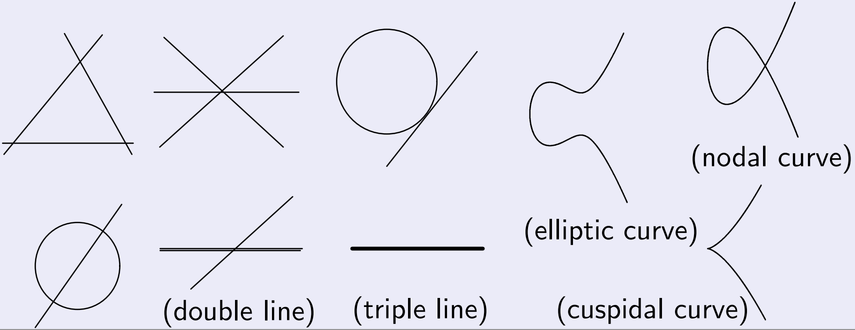
(2) A satisfies (G2) ($A = \mathcal{A}(E, \sigma)$) $:\Longleftrightarrow \exists (E, \sigma)$ s.t. $R = \{f \in k\langle x_1, \ldots, x_n \rangle_2 \mid f(p, \sigma(p)) = 0, \forall p \in E\}$.

(3) A: *geometric* $:\Longleftrightarrow$ A satisfies (G1), (G2) and $A = \mathcal{A}(\mathcal{P}(A))$.

Theorem by ATV

Theorem ([Artin-Tate-Van den Bergh, 1990])

$\forall A$: 3-dimensional quadratic AS-regular algebra, A: *geometric*.
Moreover, when $\mathcal{P}(A) = (E, \sigma)$, $E = \mathbb{P}^2$ or a cubic divisor in $\mathbb{P}^2$ as follows.



Normalizations of varieties

By using the normalization of a variety, we determine the defining relations of Type CC and Type NC 3-dimensional quadratic AS-regular algebras. (these algebras correspond to cuspidal and nodal cubic curve in $\mathbb{P}^2$).

Proposition

E: a irreducible variety, π: $\tilde{E} \longrightarrow E$: the normalization of $E \Longrightarrow$ $\forall \sigma \in \operatorname{Aut} E$, $\exists^1 \varphi \in \operatorname{Aut} \tilde{E}$ such that $\sigma \circ \pi = \pi \circ \varphi$.

$$\begin{array}{ccc} \tilde{E} & \xrightarrow{\pi} & E \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\sigma} \\ \tilde{E} & \xrightarrow{\pi} & E \end{array}$$

Type CC $E = \mathcal{V}(x^3 - y^2 z) \Longrightarrow \pi : \mathbb{P}^1 \longrightarrow E$;
$\pi(a : b) := (a^2 b : a^3 : b^3)$: the normalization of E.

Type NC $E = \mathcal{V}(x^3 + y^3 + xyz) \Longrightarrow \pi : \mathbb{P}^1 \longrightarrow E$;
$\pi(a : b) := (a^2 b : ab^2 : -a^3 - b^3)$: the normalization of E.

Main Theorem 1

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Type CC

$$\begin{aligned} A &= \mathcal{A}(E, \sigma_r) \\ &= k\langle x, y, z\rangle / \left(\begin{array}{c} -3r^2x^2 + 2r^3xy + xz - zx - 2rzy, \\ xy - yx + ry^2, \\ -3rx^2 - r^3y^2 + yz - zy \end{array} \right), \end{aligned}$$

where $\sigma_r(x : y : z) = (rxy + x^2 : xy : r^3xy + 3r^2x^2 + 3ryz + xz)$ $(r \neq 0, 1)$. Moreover, $\forall r, r' \neq 0, 1$, $\mathcal{A}(E, \sigma_r) \cong \mathcal{A}(E, \sigma_{r'})$.

Type NC

- Case 1

$$A = \mathcal{A}(E, \sigma_{1,s}) = k\langle x, y, z\rangle / \begin{pmatrix} xy - syx, \\ (s^3-1)x^2 + s^2zy - syz, \\ (s^3-1)y^2 + s^2xz - szx \end{pmatrix},$$

where $\sigma_{1,s}(x\colon y\colon z) = (sxy\colon s^2y^2\colon (s^3-1)x^2 + s^3yz)$ $(s^3 \neq 0, 1)$.

  - $A = \mathcal{A}(E, \sigma_{1,s})$, $A' = \mathcal{A}(E, \sigma_{1,s'}) \Longrightarrow A \cong A' \iff s' = s^{\pm 1}$.
  - $\mathrm{GrMod}\, A \simeq \mathrm{GrMod}\, A' \iff s'^3 = s^{\pm 3}$.

- Case 2

$$A = \mathcal{A}(E, \sigma_{2,t}) = k\langle x, y, z\rangle / \begin{pmatrix} txz + (1-t^3)yx - t^2zy, \\ tzx + (1-t^3)xy - t^2yz, \\ y^2 - tx^2 \end{pmatrix},$$

where $\sigma_{2,t}(x\colon y\colon z) = (ty^2\colon t^2xy\colon (1-t^3)x^2 + yz)$ $(t^3 \neq 0, 1)$.

  - $\forall t, t'$, $\mathcal{A}(E, \sigma_{2,t})) \cong \mathcal{A}(E, \sigma_{2,t'}))$.

Calabi-Yau algebras and conjecture

Definition ([Ginzburg, 2007])

C: d-dimensional Calabi-Yau algebra $:\Longleftrightarrow$

(i) $\mathrm{pd}_{C^{\mathrm{e}}} C = d < \infty$, ($C^{\mathrm{e}} := C \otimes_k C^{\mathrm{op}}$: the enveloping algebra of C)

(ii) $\mathrm{Ext}^i_{C^{\mathrm{e}}}(C, C^{\mathrm{e}}) = \begin{cases} C & (i = d), \\ 0 & (i \neq d). \end{cases}$ (as C^{e}-module)

Conjecture

$\forall A$: 3-dimensional quadratic AS-regular algebra, $\exists C$: a Calabi-Yau AS-regular algebra s.t. $\mathrm{GrMod}\, A \cong \mathrm{GrMod}\, C$.

Using the defining relations in Theorem 1 for Type CC and Type NC and for another case in [Matuzawa-Kim], and using a twist of superpotential in the sence of [Mori-Smith, 2016], we show that this conjecture holds in most cases.

Theorem 2

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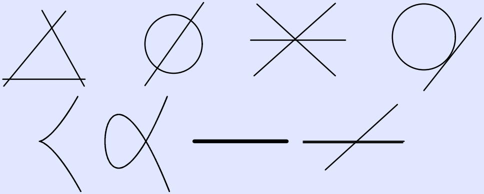
E: $\mathbb{P}^2$ or a cubic divisor in $\mathbb{P}^2$ as follows:



$A = \mathcal{A}(E, \sigma)$: 3-dimensional quadratic AS-regular algebra corresponding to E and $\sigma \in \operatorname{Aut} E \Longrightarrow \exists C$: a Calabi-Yau AS-regular algebra s.t. $\operatorname{GrMod} A \cong \operatorname{GrMod} C$.